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# of
# ELECTRIC MACHINERY

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GABRIEL KRON

*Consulting Engineer*
*General Electric Company*

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*To*

SELDEN B. CRARY

# PREFACE

In this book equivalent circuits (*stationary* electric-circuit models) are developed for all *rotating* electric machines and groups of machines by means of a unified physical picture, without any mathematical analysis. The behavior of all types of machines or groups of machines is represented in the form of equivalent circuits under all operating conditions that may be expressed in terms of constant, sinusoidal, or sums of sinusoidal currents and speeds. These operational conditions embrace, first of all, the usual steady-state, constant-speed operations, including the case in which time harmonics or space harmonics (or both) arise, or the applied voltages are unbalanced, or the impressed frequencies are variable. Moreover, instantaneous and sustained polyphase or single-phase short circuits, sudden load variations, and self-excitations are also considered.

Although the equivalent circuits developed are valid for hunting problems also (small oscillations superimposed upon steady rotation), the detailed study of hunting is not undertaken here but is left for future consideration. The present book lays only the foundation for the eventual systematic treatment of stability and hunting of individual and groups of machines and transmission systems by the same unified physical picture and without the aid of equations of performance.

Practically all types of a-c rotating machines used in the power industry are represented. Unbalanced induction machines (including the capacitor and shaded-pole motors with space harmonics), salient-pole synchronous machines, single-phase alternators, a-c commutator motors, and several interconnected machines running at the same speed or at different speeds are studied in complete detail. Because of their importance, the synchronous machine and the polyphase induction motor are examined under greater variety and more detailed operating conditions than other types of machines.

The equivalent circuits show all fundamental and harmonic currents and fluxes in every winding of the machine, and also the constant and harmonic torques developed by the rotor. For most machines both the cross-field and revolving-field equivalent circuits are given, often in several simplified forms. Such complex problems as eddy currents in solid rotors, turn-to-turn short circuit of a double-winding generator,

currents flowing in each bar of a non-uniform and non-symmetrical amortisseur winding, eccentric rotors, and space and time harmonics due to the winding distribution and slot openings of a polyphase induction motor are investigated in detail.

*All equivalent circuits are established without the use of a single equation of performance.* The reasonings are based upon a generalization of the equations of a stationary two-winding transformer only, aided by a physical picture of the flux paths inside the machines and by the observable interconnection of the windings. It is proved in the Epilogue that with certain precautions the equations of all rotating electric machinery may be reduced to Ohm's law $\mathbf{e} = \mathbf{Zi} = (\mathbf{R} + \mathbf{jX})\mathbf{i}$ (with a symmetrical $\mathbf{Z}$) under all sinusoidal operating conditions, provided that the equations correspond to the dynamical equations of Lagrange and their generalizations, and provided that the correct reference frame is used.

The word "tensor" is not used in the text (except in Prologue and Epilogue), but the method of attack follows strictly the basic tensorial equations of electrodynamics and the tensorial reasoning employed in their reduction to particular machines. In the present book a tensor (a physical entity) is represented for any particular machine, not by a set of mathematical symbols arranged in a matrix (mathematical model), but by a set of electrical symbols (electric-circuit model). The parallelism between the mathematical and electrical models is so close and complete that the equivalent circuits themselves may be employed to write down, *by mere inspection*, the equations of performance of any type of machine, not only under *sinusoidal* operating conditions, but also—by a slight change in symbolism—under *transient* and accelerating operation.

The physical analysis itself follows the usual analytical procedure used by the author in his other works * on electric machinery. That is, first the equivalent circuit of a prototype, the so-called "primitive" machine, at rest is established, afterwards the effect of rotation is included, and finally *the equivalent circuit of each type of machine is found from that of the primitive machine by means of a transformation.*

*No acquaintance with the author's other works is needed.* The book is self-contained, though some acquaintance with the underlying philosophy of the tensorial method should help. The only prerequisite for the understanding of the book (beside some familiarity with the elementary

* "The Application of Tensors to the Analysis of Rotating Electrical Machinery," *General Electric Review*, Schenectady, N. Y., 1938; *Tensor Analysis of Networks*, John Wiley & Sons, New York, N. Y., 1939; *A Short Course in Tensor Analysis for Electrical Engineers*, John Wiley & Sons, New York, N. Y., 1942.

theory of machines) is an earnest desire to learn it. However, the equivalent circuits may be copied and used for the solution of practical problems without understanding how they have been derived.

A word of caution may be injected. The method of reasoning in this book is wholly deductive; all specific examples are derived as special cases of one general principle. Now, to many minds such a reasoning is totally strange; many people like to examine first each special case in detail and pass from the simpler case to the next slightly more complex one in easy stages. Engineers preferring such inductive methods should approach this book with an open frame of mind and proceed with a receptive attitude, in order to benefit from the author's deductive reasonings.

These equivalent circuits have been developed by the author during the last decade in connection with his everyday engineering tasks. Some of the circuits have already been published in the *General Electric Review* and in the *Transactions of the American Institute of Electrical Engineers*, but most of them appear in this book for the first time. Of course, some of the simpler circuits, such as those of the single or interconnected polyphase induction motors and the single-phase induction motor, have been known for decades. Nevertheless, even such an elementary and important equivalent circuit as that of the salient-pole synchronous machine running at synchronous speed was not available until recently.

In this book even the well-known equivalent circuits have been generalized to be valid for *variable-frequency* impressed voltages also. The introduction of variable-frequency $f$ operation allows a study of self-excitation of systems and simplifies the interconnection of the equivalent circuits when machines of various types run at different speeds. Moreover, all equivalent circuits become models for writing down outright not only the steady-state but also the *transient* and *accelerating* equations of performance of any complex system, by simply replacing the variable-frequency symbol $f$ everywhere by $-jp = -j(d/dt)$. The variable-frequency equivalent circuits are ideally suited to establish the transient transfer functions of groups of machines, needed in stability studies.

The absence of mathematics in the establishment of the equivalent circuits should not be construed as an attempt to belittle the importance of a strictly mathematical development of equivalent circuits. As a matter of fact, practically all equivalent circuits given here have originally been developed mathematically first, and only after the availability of many examples did the organic unity of all circuits become apparent. With unusual windings, or with more complex interconnection, or with certain special design requirements, the avoidance of mathematics may

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become quite a nuisance, and in any doubtful case a mathematical set of equations must serve as the final court of authority for the correctness of the model. Example of one such mathematical development of the constants of a circuit is given in Appendix 2 for the shaded-pole motor.

The first half of the book (contained in seven chapters) forms a consistent group and covers in detail much of the material on induction, synchronous, and commutator machines that advanced textbooks touch upon or merely allude to. Much of the subject matter of the remaining five chapters appears by conventional analytical tools so highly advanced that they are not to be found in any textbook, or in periodicals even, except perhaps in sketchily traced outlines. Beginning with the seventh chapter, on commutator machines, parts of chapters or even entire chapters may be left out to suit the special interest of the reader. Several numerical examples are also included.

The systematic development of the equivalent circuits has been undertaken at the request of S. B. Crary. The author is indebted also to C. Concordia and P. L. Alger for many conversations. Thanks are due to Michael Temoshok for reviewing the book from the point of view of the interested engineer.

GABRIEL KRON

*Schenectady, N. Y.*
*August 1951*

# CONTENTS

# PROLOGUE

## The Philosophy of Equivalent Circuits

### WHAT IS AN EQUIVALENT CIRCUIT?

An equivalent circuit, consisting of stationary electric-circuit elements, is intended to be a stationary model of a rotating machine. It is intended to represent the magnitudes and frequencies of the various currents, fluxes, and torques existing in each winding of the rotating machine, or differ from them, if at all, merely by a simple linear relation.

Excluding those of the polyphase induction motor and its derivatives, the equivalent circuits of rotating machines occasionally appearing in the literature satisfy only one of the above several requirements of what a true model should be. These so-called equivalent circuits usually give the magnitude of the line current only, and sometimes they may give several of the currents. But they almost never represent the torque in the correct manner as the product of a flux density and a current (as it is actually defined by the equations of dynamics). The torques are supposed to be calculated either by subtracting the losses from the input power or with the aid of auxiliary formulas developed. The brute-force equivalent circuits usually do not show the fluxes linking each winding, or the currents in each winding, or their frequencies.

All equivalent circuits given in this volume are true models of the rotating machine in question within the limitations defined by the basic assumptions themselves. The circuits to be developed define not only the magnitudes but also the frequencies of the currents flowing in each winding, the resultant flux densities (or flux linkages) linking each winding, and the impressed voltages, in addition to defining the component and resultant torques developed by the rotor.

Although the primary incentive for the development of the equivalent circuits was the availability of a-c network analyzers for their solution, the author has found that the ease of solution of a set of equations is outweighed by far more important considerations. The simpler physical picture offered by a true model equivalent circuit (as opposed to a trick

circuit) not only helps the engineer's understanding but also enables him to attack more complicated problems that he would otherwise hesitate to approach analytically. Space harmonics and time harmonics, single-phase unbalanced problems, and the hunting of rotating machines are easily visualized and analyzed if equivalent circuits are available. In the study of several interconnected machines it is much easier to interconnect their equivalent circuits than to write an analytical expression for the combined system.

## CIRCUIT MODELS OF ROTATING MACHINES

Engineers have developed several types of models to visualize, understand, and predict the behavior of rotating electric machinery. The more important models are as follows:

1. Physical pictures.
2. Mathematical models (equations).
3. Stationary electric networks (equivalent circuits).
4. Vector and locus diagrams.

The primary purpose of the present book is to develop a systematic theory of electric machinery in terms of equivalent circuits only. The circuits in turn may be used by the engineer to solve the performance of the machine either by numerical methods or with the aid of a network analyzer. Repeated experience (and a mathematical proof given in Appendix 2) has demonstrated conclusively that *it is always possible to represent the operation of any rotating machine or group of machines by means of stationary networks*, provided that two conditions are satisfied:

1. The machines operate at constant speed or with small oscillations superimposed upon a constant speed (hunting).
2. The currents may be expressed as sums of sinusoids in time.

Space harmonics and time harmonics, commutator brushes and windings at an angle, unusual windings like non-uniform and non-symmetrical amortisseur bars, and the interconnection of individual windings or of entire machines are no barriers to the electric-circuit model representation.

For induction and synchronous machines only positive or negative resistances, inductances and voltage generators are needed for representation. Commutator machines and the interconnection of machines may involve phase shifters, ideal transformers, and even amplifiers.

When only three types of circuit elements (resistors, inductors, and capacitors) are available in connection with an a-c network analyzer, it is always possible to transform the given equivalent circuits into a more easily manipulative form for use on the analyzer. It is not the

purpose of the present book to go into the details of such representation. The book restricts itself to setting up on paper stationary networks that simulate the performance of a rotating machine in as great detail and in as close an analogue as possible. Variations in the given circuits for other purposes will easily suggest themselves.

## DIFFERENCE BETWEEN A ROTATING MACHINE AND A STATIONARY NETWORK

When the voltage equations of the form $e = zi$ are written for any *stationary* electric network, they have the important property that all mutuals are reciprocal (the impedance matrix $Z_{\alpha\beta}$ is symmetrical). However, in the voltage equations of rotating electric machinery the voltages induced in two meshes are usually not reciprocal. A current in winding $a$ may not generate a voltage in winding $b$ even though the current in winding $b$ does generate a voltage in winding $a$. (The impedance matrix $Z_{\alpha\beta}$ is not symmetrical.)

Now it is usually possible to take any set of equations with non-reciprocal mutuals and to change them into a set with reciprocal mutuals by *brute force*, that is, by a method that bears no relations to the physics of the problem. In such cases the physical picture inherent in the original equations has usually been distorted or ruined during the forced manipulation.

However, even though the original set of equations has been brought to a reciprocal form ($Z_{\alpha\beta}$ is symmetrical), it still does not follow that a stationary network may be established for it. In general, a set of $n$ equations with reciprocal mutuals can be represented by an $n$-winding transformer only if all mutuals have the same sign. When the mutuals have different signs either the model is non-existent or some (brute-force) representation may be accomplished with the aid of ideal transformers.

All equivalent circuits in this book are free of $n$-winding transformers or ideal transformers even though they involve often from eight to thirty-two coupled meshes (only in some commutator machines is a two-winding actual transformer—not ideal transformer—introduced).

Nevertheless, the main aim in the construction of equivalent circuits is not the avoidance of transformers, amplifiers, and other artifices but the *preservation in the stationary model of all physical attributes of the original rotating machine* and the automatic establishment of the model without the aid of any mathematical derivation. Also, the aim is to use identical reasoning for all types of machines and groups of machines. From this point of view, the services of any device are legitimate that help accomplish that purpose.

## TRANSIENT AND ACCELERATING EQUATIONS OF PERFORMANCE

The equivalent circuits to be developed cannot be set up on a transient analyzer to establish *exact* solutions of transient problems, such as sudden short-circuit or acceleration problems. (In the last chapter, equivalent circuits are given for the *approximate* solution of sudden short circuits and sudden load variations.) However, if special *rotating* devices are introduced at certain strategically located points in the equivalent circuits, they can be used to solve transient problems, also, with the aid of an analyzer. Such generalizations are not considered in the present volume.

Nevertheless, all equivalent circuits given have been so organized that they may be used outright to *write down*, not only the sinusoidal, but also the transient and accelerating equations of performance of any rotating machine or group of machines by simple inspection. After a minor manipulation to be shown, all equations assume the form given by rigorous analytical derivations that employ the basic equations of electrodynamics as a starting point.

## THE TENSORIAL POINT OF VIEW

This book undertakes an unusual experiment. Although the equivalent circuits derived here for a single machine or for a group of machines do give outright the equations of performance even under transient and accelerating conditions, nevertheless, *all equivalent circuits are established here without writing down at any time a single equation of performance.* All developments are based upon a familiarity with the equations of a two-winding *stationary* transformer only.

The steps in going from a *two-winding* transformer to an *n-winding* machine or groups of machines, on one hand, and the steps in going from the model of a *stationary* transformer to the model of a rotating machine with rotating reference frames, on the other, will be based upon a deductive mode of thought that has been slowly influencing the scientific attitude during the last century and a half. That deductive method of attack will be called here—for lack of a better name—the "tensorial" point of view. Perhaps the expression "unified" or "generalized" point of view might serve the same purpose, but the word "tensor" defines the scope and limitations of this mode of thinking more specifically.

## THE TENSOR AS A GEOMETRICAL TOOL

Although tensor analysis is usually thought of as a *mathematical* discipline, primarily it is a tool for *geometrical* thinking. It has been developed gradually during the last century and a half to supply an

urgent need to deal in a uniform manner with several complex notions that arose at isolated places but tended to converge in the same directions. The higher than three-dimensional spaces indicated by Grassman, on one hand, and the non-Euclidean (curved) spaces envisaged by Bolyay and Lobachevsky, on the other, early demanded a more easily manageable mathematical symbolism. The matrices of Cayley, the vector analysis of Gibbs and Heaviside, and the quaternions of Hamilton were aimed toward simpler symbolism; however, the growing mathematical theories of invariants and groups have soon shown the inadequacy of each of these developments.

The organization of these widely scattered notions into one tool was undertaken by Ricci under the name "tensor calculus" and was continued by Levi-Civita, who called the same tool "absolute calculus." Since the beginning of this century this organized tool has grown to immense proportions in the hands of Weyl, Cartan, Veblen, Kawaguchi, and Schouten, to mention only a few original workers in the field of affine, projective, and conformal geometries.

## THE TENSOR AS A PHYSICAL TOOL

Because of the nebulous dividing line existing between geometry and physics, tensors were applied to pure physics in the very early days. The dynamics of moving bodies has been expressed in terms of the motion of a point in an $n$-dimensional non-Euclidean (or Riemannian) space. The basic equations of three-dimensional fields, so powerfully expressed in terms of the gradient, divergence, and curl of vector analysis, have also been rewritten in the language of tensors.

It should be mentioned here that, in conventional field problems, because of the existence of only three variables, the gain by the use of tensor analysis is slight (compared to vector analysis). Lately in the complex problems of *turbulent* flow, however, tensors have indicated their greater versatility.

Similarly, the use of tensors in the reinterpretation of the dynamical equations of Lagrange has only been of academic interest, chiefly because of the narrow interpretation placed upon the idea of "transformation." In order to employ tensorial methods in the systematic formulation and solution of *practical* dynamical problems, it is absolutely necessary to introduce the theory of $n$-dimensional subspaces immersed into space with still higher dimensions,* and the concept of "interconnection" of spaces with a different number of dimensions.

* "Kron's Method of Subspaces," by Banesh Hoffmann, *Quarterly of Applied Mathematics*, Vol. II, No. 3, October 1944.

Of more interest to the electric-power engineer is the application of tensorial methods to pure electrodynamical problems. The general theory of relativity of Einstein and especially the various "unified field theories" proposed by Einstein,* Weyl, Schouten, and others to unify gravitational and electromagnetic phenomena have introduced peculiar types of spaces with "torsion" that surprisingly enough fit almost perfectly the theoretical foundations of rotating electric machinery.

The analogy is due to the *coexistence* in rotating electric machinery of both electric and mechanical energies and to their mutual interactions considered from reference frames that may have an accelerated rotation with respect to the electromagnetic field and the material bodies. Of course, the mathematical analogy is only formal, since mechanical energy is not the same as gravitational energy. The formal analogies may be extended, though, even to the theories of the elementary particles of nuclear physics.†

## THE TENSOR AS AN ENGINEERING TOOL

Many problems of the engineer involve the analysis of interrelated physical phenomena, which requires the manipulation of a large number of variables, and, since the algebra and calculus of tensors have been expressly created for problems with many variables, *tensor analysis is an engineering tool par excellence.* Although geometers have shown, in passing, that the basic equations of dynamics and three-dimensional field phenomena may be expressed in the language of tensors, no physicist or mathematician has as yet taken the trouble to apply the tensorial reasoning to the *systematic formulation* of practical problems. Apparently it is up to the engineer to initiate and pursue that undertaking.

Although the author has made a start in employing tensorial reasoning in the formulation of *mechanical* engineering problems, such as the interconnection of beams, each with twelve degrees of freedom,‡ and the vibrations of polyatomic molecules § control systems in steam-turbine governing system,|| the author's main interest centers on the introduc-

* See, for instance, *The Meaning of Relativity*, by A. Einstein, Princeton University Press, 1950.

† Banesh Hoffmann, "Kron's Non-Riemannian Electrodynamics," Einstein's 70th birthday commemorative issue of the *Reviews of Modern Physics*, Vol. 21, No. 3, pp. 535–540, July 1939.

‡ "Tensorial Analysis and Equivalent Circuits of Elastic Structures," *Journal of the Franklin Institute*, Vol. 238, pp. 400–442, December 1944.

§ "Electric Circuit Models for the Vibration Spectrum of Polyatomic Molecules," *Journal of Chemical Physics*, Vol. 14, No. 1, pp. 19–34, January 1946.

|| "Tensorial Analysis of Control Systems," *Journal of Applied Mechanics*, Vol. 15, pp. A107–124, June 1948.

tion of tensorial concepts and reasonings into practical *electrodynamical* problems, in particular into the analysis of electric-*power* networks and power machinery.

The present book differs from the other works of the author on tensorial analysis only in the symbolism employed and not in the method of reasoning. Wherever in the other books the symbols $jx$ and $r$ appear, in this book the drawings of an inductor and a resistor are shown. Wherever in the other books the *sum* of two symbols appear as $r + jx$, here a resistor and an inductor are connected in series. The symbol of rotation, $\epsilon^{j\theta}$, is replaced here by a phase shifter, the current by an arrow, an impressed voltage by a circle, etc. In spite of the absence of mathematical formulas, *this present work is an example of the application of tensorial reasoning.*

## GENERALIZATION POSTULATES

The steps made here in arriving at the final *equivalent circuit* and the steps made elsewhere in arriving at the final *equations of performance* of any particular machine follow identical patterns. That pattern may best be illustrated in terms of three so-called "generalization postulates." These postulates act as unfailing beacons among a labyrinth of intermingled phenomena and offer definite clues on how to unravel a seemingly hopeless maze. Although standard textbooks on tensor analysis never express these postulates, the postulates are nevertheless present in an implied manner and occasionally are stated in a less explicit but more mathematical form.

*The argument of the present book is based chiefly upon the third generalization postulate, which formulates that all rotating electric machines differ from stationary electric networks only by a set of dimensionless scalars—the so-called "absolute" frequencies—whose values can be determined simply by an inspection of the terminals, shafts, and reference frames protruding from the machine, viewed as a closed box.*

Each of the postulates will be formulated at first geometrically, then mathematically, and finally "electrically," namely in terms of electric networks. The precise manner of statement of these postulates is of no importance, as the postulates represent only a method of thinking and as such they must be kept fluid to fit the everchanging variety of physical problems.

## A "PRELIMINARY" GENERALIZATION POSTULATE

In stating any theorem, for instance, the Pythagorean theorem, the geometrician does not attempt to describe all such possible relations numerically as $5^2 = 3^2 + 4^2$, etc. Instead, he replaces all possible right-

angled triangles by one symbolic triangle with sides $a$ and $b$ and writes for the hypotenuse $c^2 = a^2 + b^2$. It can be postulated that:

*An infinite variety of arithmetic equations may be replaced by one algebraic equation of the same form if each numeric is replaced by an appropriate letter.*

Similarly, Ohm's law is stated not specifically for every possible electric circuit but only symbolically as $e = Zi$ for one symbolic impedance.

Of course, this generalization postulate has become second nature to the engineer, who hardly ever stops to think of it as such. Nevertheless, it took thousands of years of development to make the change in symbolism from $6 = 2 \times 3$ to $c = ab$. Also, it should be mentioned specifically that, in spite of the extensive use of algebraic symbolism, in any engineering problem a definite amount of numerical work must be performed at the end of the analysis.

## THE FIRST GENERALIZATION POSTULATE

To deal with a large number of variables without undue mental effort, the geometrician replaces the conventional two- and three-dimensional space by a hypothetical $n$-dimensional space and generalizes the theorems of ordinary space to apply to his new space. He still talks about the "distance between two points" and the "angle between two lines," even though all these concepts extend in an $n$-dimensional space, where $n$ is any large number, including infinity. The mathematician similarly postulates that:

*The $n$ algebraic equations describing a physical system with $n$ degrees of freedom may be replaced by a single equation having the same form as that of a single unit of the system, if each letter is replaced by an appropriate $n$-way matrix ($n$-dimensional matrix).*

In setting up the equivalent circuit of a rotating machine having several layers of windings on the stator and rotor, the theory applicable to two-winding transformers will be immediately generalized to an eight-mesh network instead of extending laboriously the basic theory first to three meshes, then to four meshes, etc. Of course, this immediate generalization requires the proper spatial arrangement of resistors and inductors in a network which corresponds closely to the spatial arrangement of the impedances $r + jx$ in the form of a matrix, on one hand, and to the winding arrangement in the rotating machine itself, on the other.

## THE SECOND GENERALIZATION POSTULATE

Let a geometrical configuration be given, for instance, two points lying in an $n$-dimensional space. One of the first questions of the geom-

etrician is: "What are the properties of the configuration that will remain the same ('invariant') no matter what reference frames are used in describing them?" For example, the distance between the two points remains the same in rectangular, cylindrical, elliptical, or any of the infinite variety of reference frames along which the distance may be expressed. If a formula for the distance between two points is available in any one reference frame, then the same formula may be "transformed" in a routine manner to be applicable to any other reference frame. The mathematician would postulate this invariant property of the distance in the following manner:

*If the matric equation of a particular physical system is known, the same equation is valid for a large number of physical systems of the same nature, provided that each n-way matrix in the equation becomes an appropriate tensor (or rather a geometric object), that is, if each n-way matrix becomes endowed with a definite law of transformation.*

If the equivalent circuit of any particular rotating machine (with a large enough number of windings) has once been established, then the equivalent circuit of all other rotating machines may be derived from it by an appropriate "transformation" that leaves the spatial position of all resistors and inductors undisturbed and changes only their interconnection.

That particular machine whose equivalent circuit gives that of any other machines with the least effort and disturbance is called here the "primitive" machine. It only has theoretical existence on paper. The equivalent circuits of all other industrial machines are derived from it by open-circuiting some meshes, changing some number of turns, inserting phase shifters, etc., *without disturbing the configuration of the inductors and resistors.*

It cannot be sufficiently emphasized that the whole key to the book lies in the last phrase, "without disturbing the configuration of the inductors and resistors." When a tap or a set of brushes or slip rings is attached to the terminals of a machine, the magnetic circuits and the windings in the slots remain unchanged. Consequently, their model symbolism also must remain unchanged.

## THE THIRD GENERALIZATION POSTULATE

Let two points lie in the plane of the paper (a two-dimensional Euclidean space) and let the equation of the shortest line passing between the two points be given. Suppose now that the plane is changed to a sphere or an ellipsoid. The question arises whether the equation valid for the plane will be valid for the curved surfaces also. The mathematician answers this question by the following postulate:

*The equations describing the properties of n-dimensional Euclidean spaces become applicable to n-dimensional non-Euclidean spaces also, if all "ordinary" derivatives in the equations (such as $dA/ds$) are replaced by so-called "absolute" derivatives (or "covariant" derivatives $\delta A/ds$) derived from the former by a routine procedure.*

It has been shown by the author (see Epilogue) that the equation of any *rotating* electric machines with rotating reference axes is the same as that of a *stationary* network containing resistors and inductors, provided that all ordinary time derivatives $L\, di/dt$ are replaced by absolute time derivatives $L\, \delta i/dt$.

## THE ABSOLUTE FREQUENCIES

When in stationary networks ordinary time derivatives $di/dt = pi$ are replaced by sinusoidal phenomena, then $p$ becomes $j\omega = j2\pi(f)$ where $f$ is the frequency of the current $i$. In analogy to it, when in rotating machinery all *absolute* time derivatives $\delta i/dt$ are replaced by sinusoidal quantities, then $\delta/dt$ becomes $j\Omega = j2\pi(f \pm v_1 \pm v_2)$, where $v_1$ is the velocity of the particular winding and $v_2$ the velocity of the particular reference frame. Hence the reactance of each winding $pL = j\omega L = j2\pi fL$ in a rotating machine becomes $(\delta/dt)L = j2\pi(f \pm v_1 \pm v_2)L$.

The expression in the parentheses $f \pm v_1 \pm v_2$ is called the "absolute" frequency, and it represents the actual frequency of current in a particular winding measured along a reference frame rigidly connected to the winding.

That is, *the only effect of the rotation of the conducting or magnetic materials or of the reference frame is to change the single frequency $f$ impressed upon each winding to a so-called "absolute" frequency $f \pm v_1 \pm v_2$, which is different for each winding. Hence, if the resistance and impressed voltage of each winding are divided by the absolute frequency, all rotating machines become stationary networks, in each winding of which currents of only unit frequency flow.*

It is this third generalization postulate upon which the present book is based. By allowing the formulation of the performance of all rotating electric machinery in terms of "absolute" (or "covariant") time derivatives and replacing the latter by sinusoidal time quantities, the theory of all rotating electric machines thereby reduced automatically to the theory of stationary networks.

## SUMMARY OF THE GENERALIZATION POSTULATES

1. The first generalization postulate establishes the equivalent circuit of the primitive machine *at standstill*, containing as many meshes on the stator and rotor as any single rotating machine might likely possess.

2. The second generalization postulate changes the equivalent circuit of the primitive machine to that of any industrial machine or group of machines by a routine procedure. The change involves only the manipulation of the impedanceless connections. (The machines are still at standstill.)

3. The third generalization postulate changes the equivalent circuit of a machine or a group of machines at standstill to one having uniform rotations or small oscillations superimposed upon uniform rotations.

The change involves only the division of the resistance and sinusoidal impressed voltages by a set of dimensionless scalars, the so-called "absolute" frequencies, that are equal to the actual frequencies of currents in the conductors.

## THE ABSENCE OF MATHEMATICS

*Each of these steps is established in the book without any mathematical derivation or manipulation. Each of these steps is accomplished by mere inspection of the actual machine.* In particular:

1. The equivalent circuit of the primitive machine *at standstill* is established by the inspection of the magnetic circuits. The inductors are arranged in the same manner as the reluctances of the flux paths are located, and the resistors are arranged in the manner in which the windings link the magnetic paths.

2. The equivalent circuit of the primitive machine *in rotation* is found by inspecting the speeds of the rotor ($v_1$), the reference frame ($v_2$), and the impressed voltage wave ($f$).

3. The equivalent circuit of any particular machine is found by inspecting the manner of interconnection of windings and reference frames of the actual machine. The same interconnections are performed on the equivalent circuit.

## THE LAW OF PHYSICAL MODELS

The total absence of mathematics should only emphasize to the reader the long and serious mathematical quest that preceded this unification and simplification. As a summary of the philosophy of equivalent circuits (and other types of models) of physical phenomena, it may be formulated, that:

*Algebraic or ordinary and partial differential equations may be represented by physical models only if the equations are expressed in terms of tensors.*

# 1 THE PHYSICAL MODEL

## THE PARTICLE AND WAVE POINTS OF VIEW OF ROTATING MACHINES

One way to view the performance of a rotating machine is to fix attention to the currents and voltages appearing at its *terminals*. By making suitable measurements only at the accessible terminals, it is possible to foretell the performance of any machine under all operating conditions. The rotating machine is considered an analogue of a closed box from which wires and a shaft protrude. By manipulating the visible appendages, the behavior of the dynamical system inside the box is to be predicted. Such a representation is equivalent to that of the dynamical equations of Lagrange, which consider the performance of a rotating machine as the motion of a particle in an $n$-dimensional abstract space. This point of view is not going to be maintained in the present book.

Another method of attack takes a cross-section of the machine at right angles to its axis of rotation. In the concentric layers of windings sinusoidal current-density and flux-density waves appear, which change their spatial configurations as time goes on. This cross-sectional space-wave point of view is equivalent to that of the field equations of Maxwell, and it leads to the same final equations of performance as the dynamical (particle or closed-box) method of analysis.

## GENERALIZATION OF THE MAXWELL-LORENTZ FIELD EQUATIONS

The physical theory underlying the construction of equivalent circuits will be based upon this last point of view; namely, that a *rotating electric machine is a special type of electromagnetic field, in which the distribution of electric and magnetic conducting materials and their relative motion forces the electromagnetic waves to propagate along several coupled concentric circles.* The wave motion is complicated by the fact that the couplings between the concentric circles are not merely magnetic but also material. Moreover, the material points of contact do not remain stationary but

rotate with velocities different from those of the conducting media, in which the waves actually propagate, and of the reference frames, along which the waves are viewed.

Because of the *relative motions that exist between the various electric and magnetic conductors*, not the conventional field equations of Maxwell are used as a foundation for the establishment of equivalent circuits, but their generalizations by Hertz and Lorentz to moving bodies. Furthermore, because of the *relative motions that exist between the reference frames, the points of contacts, and the conducting materials*, the Maxwell-Lorentz field equations also have to be generalized to apply to the special type of coupled wave propagation taking place in electric machinery. These extensions have been made by engineers either intuitively, without being aware of any generalization, or consciously and in a systematic manner.*

From this systematic point of view the difference between a rotating machine and a stationary machine is the existence of *relative angular velocities* between the conducting media, the points of contacts, and the reference frames. It will be found later on that *these differences in motion may be codified into a set of scalars, the so-called "absolute frequencies," whose introduction allows the replacement of a rotating machine by a stationary network* for purposes of analysis.

## CONCENTRIC SPACE WAVES

In the present theory the winding distributions are idealized by assuming that several concentric layers of windings exist on the stator and rotor, each layer consisting of a thin cylindrical sheet of parallel copper bars (Fig. 1.1 ignores their end connections). Irrespective of whether the current is alternating or direct, whether the machine is a synchronous, induction, or commutator machine, whether the rotor has an instantaneous acceleration or is running at a uniform speed, and whether the currents are transient or sinusoidal in time, at any one instant the cross-section shows the following electromagnetic quantities (Fig. 1.2):

1. In each layer of winding a current-density wave (the **I** of Maxwell's field equations) exists with several maxima and minima extending in arbitrary directions. The wave represents currents flowing into or out of the plane of the cross-section. The number of maxima and minima is equal to the number of poles of the windings.

2. Similarly, in each layer of winding a *flux-density* wave (the **B** of Maxwell) exists with the same number of maxima and minima. The fluxes are all closed curves lying in the plane of the cross-section. The

* G. Kron, "Invariant Form of the Maxwell-Lorentz Field Equations for Accelerated Systems," *Journal of Applied Physics*, Vol. 9, No. 3, pp. 196–208, March 1938.

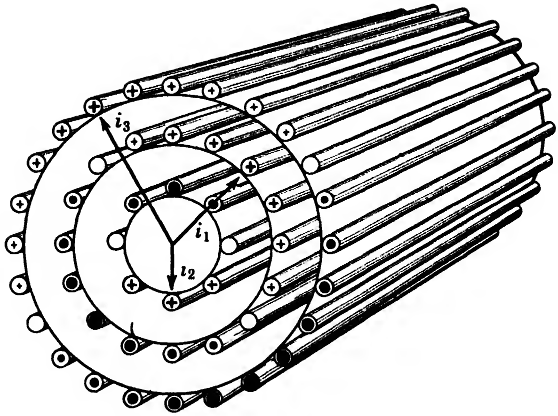



FIG. 1.1. Concentric layers of windings on a rotor.

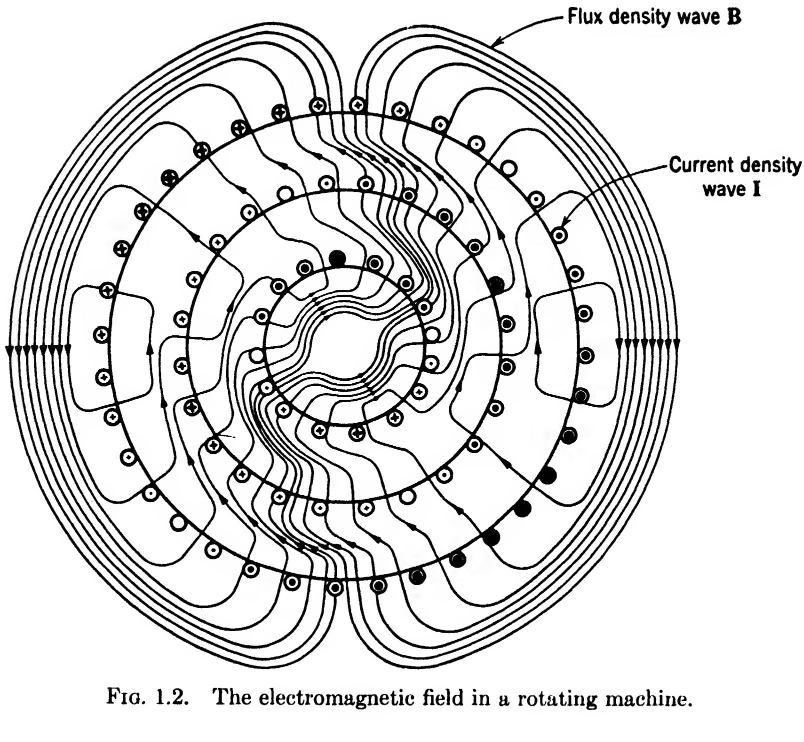



FIG. 1.2. The electromagnetic field in a rotating machine.

flux lines are assumed to cross the sheets of current densities always perpendicularly.

3. It is also possible to attribute to some of the layers of winding an impressed voltage wave (the electric-field intensity **E** of Maxwell). In analogy to **I**, they are also directed into or out of the plane of the cross-section.

As times goes on, these waves either remain frozen in space (in d-c machines) or rotate in circles without any change in magnitude or relative position (in polyphase machines). The waves may vary sinusoidally in time (single-phase machines) or may change their magnitude and position from instant to instant (transient behavior). The propagation of the maxima and minima *in space* may be prevented by removing a set of brushes or slip rings or a winding along any direction, in any of the concentric layers. The variation *in time* may be prevented by impressing a d-c voltage in any desired direction, in any of the layers.

## ADDITIONAL SPACE WAVES

In addition to the easily visualizable waves of current-density **I**, flux-density **B**, and electric-field intensity **E**, several other concepts of the field equations of Maxwell need be introduced. They are:

1. The magnetomotive-force wave (magnetic-field intensity **H** of Maxwell). It lies 90° in space from the current-density wave **I** and at each instant is equal to the latter.

2. The flux-linkage wave $\boldsymbol{\phi}$. (It is the electromagnetic vector potential **A** of the field equations of Maxwell and not the scalar potential $\phi$.) It also lies 90° in space from the flux-density wave **B** and at each instant is equal to the latter.

3. The induced-voltage vector of $d\boldsymbol{\phi}/dt$, due to the time rate of change of $\boldsymbol{\phi}$, lies in the same direction in space as $\boldsymbol{\phi}$.

4. The generated-voltage vector $\mathbf{B}p\theta$ (where $p\theta$ is the instantaneous velocity of the rotor) lies also in the same direction as the **B** vector.

5. The resistance-drop vector $\mathbf{I}r$ is in the direction of **I**.

## THE CONSTITUTIVE EQUATIONS

The presence of copper and iron in the electromagnetic field introduces two of the constitutive equations of Maxwell:

1. The presence of the magnetically conducting iron structure introduces $\mathbf{B} = \boldsymbol{\mu}\mathbf{H}$. If one of the stator or rotor structures were not salient, the permeability operator $\boldsymbol{\mu}$ would be a scalar. Because of the presence of a salient structure, the sinusoidal **B** wave is shifted in space from the sinusoidal **H** wave producing it.

2. The resistance of the conductors introduces $\mathbf{I} = \sigma\mathbf{E}$ (or $\mathbf{E} = \mathbf{RI}$). Since the windings have different resistances, the **I** and **RI** waves are not in space phase.

Strictly speaking, *the inductors and resistors of the equivalent circuits, to be constructed presently, are physical models only of the media, namely of the permeances* $\mu$ *and conductances* $\sigma$, *in which the electromagnetic field propagates.* Only when the equivalent circuit is excited do its voltages and currents become models of the electromagnetic field itself, which propagates inside the rotating machines.

The points of contacts between the concentric circles (brushes, slip rings, series connections, etc.) do not introduce any new physical entities. They only modify the relations between the entities already existing.

## TWO-DIMENSIONAL VECTORS

To simplify the analysis, the following assumptions will be made:

1. *A two-pole, two-phase machine is assumed*, having only one maximum and one minimum in each of its waves around the circumference.

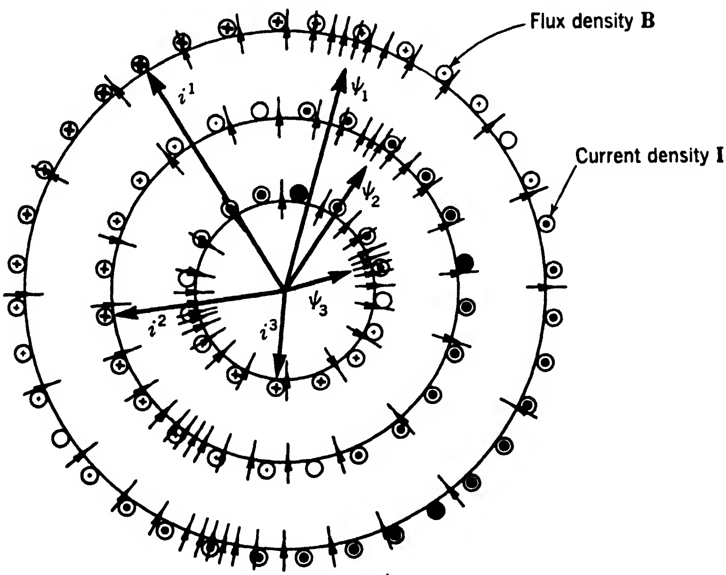



FIG. 1.3. Two-dimensional "radial" vectors.

2. The statement that all waves are sinusoidal in space is too restricted, and it should be replaced by the assumption that *the waves are sinusoidal only on the smooth structure.* For instance, in the field of a synchronous machine the current-density wave is certainly not sinusoidal

in space and certainly does not lie uniformly distributed along a circle in a thin sheet. Neither is it necessary to assume that it be so in order to obtain correct equations. These temporary thin-sheet and sinusoidal representations are made only to simplify the language of the physical picture to be developed.

When all waves have two poles, are sinusoidal in space, and are considered to be lying along thin concentric circles, it is possible to draw a vector from the center of the rotor to the positive maximum value of each circular wave, as shown in Fig. 1.3. There are as many such current, mmf, flux-linkage, flux-density, and voltage vectors as there are concentric layers of winding. In balanced polyphase machines these vectors are frozen relatively to each other and change their magnitudes and relative space relations only when the load or the speed varies.

From this pictorial point of view a rotating machine consists of a collection of actually visualizable, two-dimensional vectors **I**, **B**, **E**—as many of each as there are layers of windings. All these vectors are so-called "central" vectors, as they are all tied at one end to the axis of the rotor, being forced to rotate in the plane of the cross-section and stretch in the radial direction only.

## THE REFERENCE FRAME

To represent mathematically the two-dimensional vectors, it is sufficient to assume on each layer of winding two reference axes in space at right angles to each other (Fig. 1.4) and project the **E**, **I**, and **B** vectors upon them as $E_d$, $E_q$, $i^d$, $i^q$, and $B_d$, $B_q$. These projections are the scalar values in terms of which the performance of a machine is expressed.

In general, these reference axes may be stationary, or rotating, or may form angles other than 90°, and on the various layers of windings they may be independent of each other.

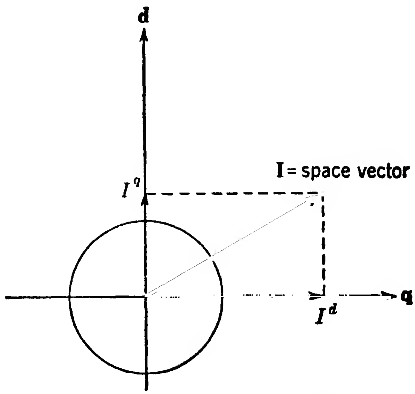



Fig. 1.4. Reference axes.

Distinction must be made between reference axes assumed in the analysis and reference axes suggested by the presence of terminals, such as brushes, slip rings, and leads. Even though the results of an analysis must be known at the terminals only, for purposes of analysis the terminals are often ignored and more convenient artificial terminals

(reference axes) are assumed. The results of the analysis are afterwards transformed to the required terminals. The previously given physical picture of a rotating machine as an assemblage of two-dimensional vectors is very effective in transforming the equations from one reference frame to another.

## ZERO-SEQUENCE VECTORS

When a three-phase machine is electrically unbalanced, on each layer of winding two sets of sinusoidal waves may be distinguished in space (assuming a two-pole machine):

1. A two-pole wave hitherto considered.
2. A six-pole wave (Fig. 1.5).

The six-pole wave is called the zero-sequence wave, and it differs in several respects from the two-pole wave:

1. It is fixed in space by the location of the three-phase terminals. As the load varies the three maxima do not shift in space.

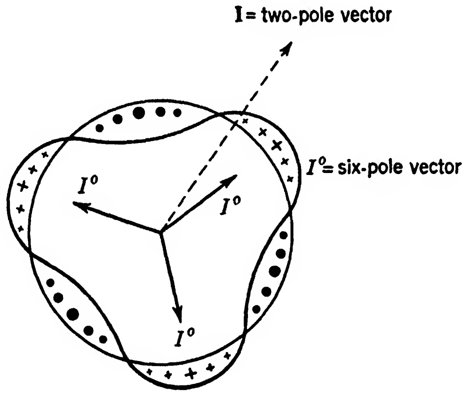



FIG. 1.5. Zero-sequence mmf wave in a three-phase machine.

2. The six-pole magnetic flux does not induce voltages in the neighboring layers and does not influence the six-pole fluxes that may exist in other layers.

3. The six-pole fluxes have no effect upon the two-pole fluxes.

Since the zero-sequence flux acts only as a leakage flux, it may be considered to belong to an additional impedance connected *outside* the rotating machine proper. *No generality is lost by omitting zero-sequence quantities altogether in the theory of rotating machines and by considering them instead as part of unbalanced three-phase stationary networks connected to the terminals of rotating machines.* The presence of zero-

sequence quantities will be ignored in the following development and will be referred to only incidentally (in Fig. 5.22), inasmuch as the theory of unbalanced three-phase *stationary* networks is not the subject matter of this book. In other words, all three-phase machines can be analyzed and will be analyzed here as if they were wound two-phase. Of course, the result of a two-phase analysis has to be transformed in order to apply to a three-phase machine. The study of such transformations is properly the subject matter of a book on three-phase unbalanced networks and is not undertaken in this book.

## NON-VISUALIZABLE VECTORS

It has been hitherto assumed that all electromagnetic quantities **I**, **B**, **E**, etc., may be visualized as waves (or radial vectors) actually existing inside the machine. The space in which each circular wave propagates is a special type of two-dimensional Euclidean space, whereas the whole rotating machine is an assemblage of several two-dimensional "centered" Euclidean spaces, interlinked and interconnected, also rotating with respect to each other.

However, there exist in rotating machines other physical quantities that cannot be visualized. One of these is the number of charges $\mathbf{q}$ that have passed through the various windings and whose time rate of change $d\mathbf{q}/dt$ gives the visualizable **I**. Another quantity is the instantaneous angular displacement $\boldsymbol{\theta}$ of the rotor conductors, moving magnetic materials, and reference axes. Their time rate of change $d\boldsymbol{\theta}/dt$ appears in the formula for "absolute" frequencies.

The role that these contrasting visualizable and non-visualizable vectors play in the construction of a basic theory of rotating machinery, is discussed in Appendix 3.

## THE PRIMITIVE MACHINE

Turning from the cross-sectional point of view to an actual two-phase, two-pole machine, the questions arise: How are the windings on it located and what are the shapes of the magnetic structures that are able to produce sinusoidal space quantities? The simplest, but still quite general, form of such a machine is as follows (Fig. 1.6):

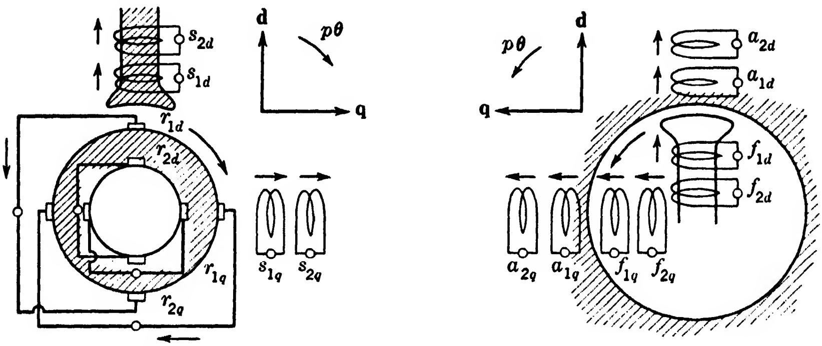
1. There exist one stationary and one rotating structure.

2. One of the structures is cylindrical; the other has two salient poles (one north and one south pole). Here, immediately, either of two special cases arises: either the stationary or the rotating structure has saliency (the latter case, Fig. 1.6*b*, occurs usually in synchronous machines; the former, Fig. 1.6*a*, in induction and commutator machines).

3. The salient structure has *two* sets of two-phase windings. One of the phases has its axis along the salient pole (direct axis); the other, at right angles to it along the interpolar space (quadrature axis). All four windings are dissimilar.

4. The smooth structure has also two sets of two-phase windings, but the two windings of the same layer are identical.

Because of the continuous nature of each layer of winding on the smooth structure, each layer may be assumed to consist of two stationary windings at right angles in space, permanently located in the same spatial




(*a*) The primitive induction machine (*b*) The primitive synchronous machine

FIG. 1.6. The primitive rotating machine.

directions as the windings on the salient structure. In Fig. 1.6*a*, even though the rotor conductors rotate, their **d** and **q** reference axes are assumed to be stationary. In Fig. 1.6*b*, the **d** and **q** axes of the stator windings are assumed to rotate with the salient pole, even though the conductors in the stator windings are stationary in space.

To summarize, the primitive machine (whose equivalent circuit is going to be established first) consists of two sets of four windings at right angles in space. The windings on the smooth structure are balanced; those on the salient structure are unbalanced.

## THE PHYSICAL REFERENCE FRAME

To return from the actual windings to the cross-sectional viewpoint, the assumption of windings along the salient structures fixes the position of the reference axes. The two axes on each layer will be at 90° from each other and will be lined up along the center of the salient pole (direct axis **d**) and at right angles to them (quadrature axis **q**). In particular:

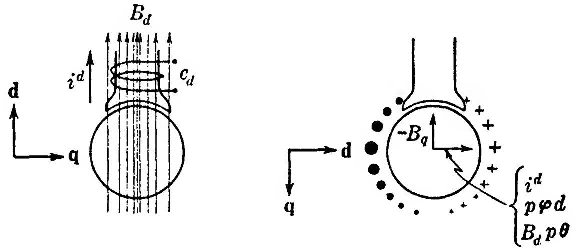
1. For induction and commutator machines the reference axes will be rigidly connected to the stator iron structure; hence they will be stationary in space. In induction machines it is customary to call the direct axis the main axis **m** and the quadrature axis the cross axis **c**.

2. In synchronous machines the reference axes will be rigidly connected to the rotor iron structure and all will rotate with the rotor.

As a second step in the analysis, another type of reference frame will be introduced for the primitive machine later on which has no physical existence, only an analytical one. That reference frame is the so-called "symmetrical component" or "sequence axes."

## TRANSITION FROM WINDING TO WAVES

Confusion often arises from the fact that it has been an accepted custom (to be followed here also) to call a current flowing in a winding wound around the salient pole (direct axis) as $i^d$, because the mmf and




Abs. curl $\mathbf{E} = -\frac{d\mathbf{B}}{dt} -$ Abs. curl $\mathbf{B} \times \mathbf{v}$ | $\mathbf{E} = -\frac{d\mathbf{A}}{dt} + \mathbf{B} \times \mathbf{v}$

i = mmf $\mathbf{H}$ | i = current-density $\mathbf{I}$

e = Abs. curl $\mathbf{E}$ | e = electric field intensity $\mathbf{E}$

(*a*) Winding directions | (*b*) Space-wave directions

FIG. 1.7. Relations between winding and space-wave directions.

flux density produced by it are directed along the direct axis. From the space-wave point of view, if **i** represents the current-density vector of this winding, then it is actually directed along the quadrature axis **q** (Fig. 1.7*a*) and logically it ought to be denoted as $i^q$.

At least two possibilities are open in interpreting the circuit equation $e = L\,di/dt$ of a winding as space waves.* The first assumption is as follows:

(*a*) **i** corresponds to the mmf wave **H** of Maxwell.

* G. Kron, "Invariant Form of the Maxwell-Lorentz Field Equations for Accelerated Systems," *Journal of Applied Physics*, Vol. 9, No. 3, pp. 196–208, March 1938.

(*b*) **e** corresponds to the space rate of change of the electric-field intensity **E** of Maxwell (curl **E**).

With these interpretations the **d** and **q** directions of both winding and waves coincide (Fig. 1.7*a*). The generalized Maxwell-Lorentz field equation corresponding to this interpretation is also given in Fig. 1.7*a*.

The second assumption is as follows:

(*a*) **i** corresponds to the current-density wave **I** of Maxwell.

(*b*) **e** corresponds to the electric-field intensity wave **E** of Maxwell.

Now the **d** and **q** directions of the space wave are at right angles to those of the circuits (Fig. 1.7*b*). The corresponding field equation is also given on Fig. 1.7*b*.

It will be shown in Chapter 8 (Fig. 8.6*b*) that the equivalent circuits to be developed presently include both these interpretations simultaneously. In particular:

(*a*) In the *horizontal* branches, **i** corresponds to an mmf wave **H**; in the *vertical* branches, to a current-density wave **I**.

(*b*) In the *horizontal* branches, **e** corresponds to a curl **E** wave; in the *vertical* branches, to an **E** wave.

## FURTHER GENERALIZATION

The expression "primitive" machine does not necessarily mean "generalized" machine, that is, the most general machine possible. Instead, it stands for some standardized structure that may be used as a starting point for the analysis and that is flexible enough to yield numerous special cases and also capable of further generalizations and complications as the need arises.

For instance, a more general assumption is that *mutual inductances* exist between the two sets of windings at right angles in space located upon the salient structure. Such an assumption will actually be needed later on, for instance, in the analysis of the shaded-pole motor (Chapter 5). For the time being, that generalization is ignored, in order not to complicate unduly the development of the basic equivalent circuits.

A more general "primitive" machine will be needed when it will be assumed that all the windings on the smooth structure also are different from each other (Chapter 11). This generalization will require the introduction of *time harmonics*, that is, several two-pole waves on each layer of winding, each wave having a different frequency (but the same number of poles).

A still further generalization will be the introduction of *space harmonics* on each layer of winding (Chapter 10). The coexistence of both space and time harmonics simultaneously is not excluded as a possibility, but

owing to the rarity of its practical application it is not considered in this book.

A different type of generalization is the replacement of a winding (producing a sinusoidal space wave) by a series of networks having no resemblance to a spatially distributed winding. This type occurs, for instance, in the study of the currents in the individual conductors of the amortisseur of a synchronous machine (Chapter 8).

In commutator machines (Chapter 7) it has to be assumed either that the flux-linkage wave $\boldsymbol{\phi}$ has no relation to the flux-density wave $\mathbf{B}$ (hitherto it has been assumed that $\boldsymbol{\phi}$ lies at right angles in space to $\mathbf{B}$ and is equal to it at each instant) or that $\boldsymbol{\phi}$ may be expressed as some linear function of $\mathbf{B}$.

The effect of saturation also forces the more general assumption that some of the inductances are not constant but are functions of the flux densities.

# 2 THE PRIMITIVE MACHINE AT STANDSTILL

## SIMPLIFYING ASSUMPTIONS

The purpose of the present chapter is to show that the configuration of the inductors in the proposed equivalent circuit is a physical model of the flux paths existing inside the rotating machine, when the latter is at standstill. Also the resistors and voltage sources of the equivalent circuit are a physical model of the windings themselves. Attention will be confined to the primitive *induction* machine, in which the stator has a salient structure. In determining the physical model it may be assumed that the rotor is at standstill. Since the two sets of windings along the direct and quadrature axes are right angles in space, no mutual inductance exists between them. Hence each set of four windings is independent of the other while the rotor is at rest, and only one set need at first be considered with its axis directed along a straight line. *All windings will also have the same number of turns.*

A further temporary simplification will consist of assuming only *two windings* in the set, one on the stator, the other on the rotor. These two stationary windings are equivalent to a two-winding transformer in which the windings are separated by an airgap. At first the well-known equivalent circuit of such a transformer will be developed step-by-step from a knowledge of its magnetic circuit. This very elementary development will be undertaken in considerable detail in order that *exactly the same* step-by-step reasoning be followed for more complicated magnetic circuits.

## STEPS IN THE ANALYSIS

The equivalent circuit of the primitive machine at standstill will be established accordingly in the following series of steps:

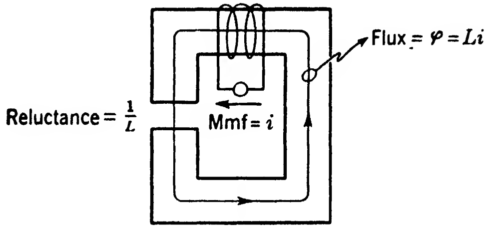
1. Two windings exist along the direct axis.
2. Four windings exist along the direct axis.
3. A similar set of four windings exists along the quadrature axis.

These steps establish the equivalent circuit of four stationary concentric layers of windings located on a salient-pole machine, assuming the

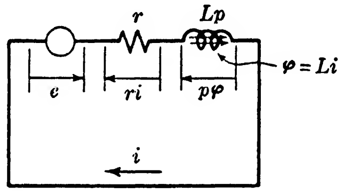
reference frames along the physically existing **d** and **q** axes. To prepare for the effect of the mechanical rotation of the rotor, one additional step will still be necessary, namely, a change of reference frames.

4. Forward and backward rotating reference frame ("sequence" axes) are introduced on each layer of winding.

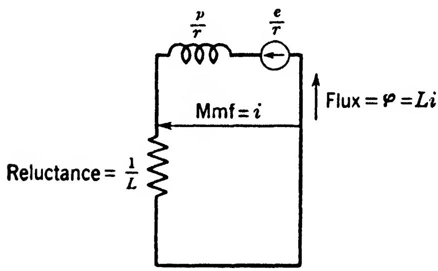
The effect of rotation is considered in a following chapter.




(*a*) Given coil and its magnetic circuit




(*b*) In terms of voltage, current, resistance




(*c*) In terms of flux, mmf, reluctance

FIG. 2.1. Two types of equivalent circuits of a single coil.

## DUAL EQUIVALENT CIRCUITS OF A SINGLE COIL

As a preliminary to the study of the two-winding transformer let a single coil wound on an iron core be considered (Fig. 2.1*a*). Its transient voltage equation is

$$e = ri + pLi = ri + p\phi \qquad 2.1$$

Its conventional equivalent circuit in terms of voltage, current, and resistance is shown in Fig. 2.1*b*. Each term of Eq. 2.1 is represented by a voltage drop in the circuit. Attention is called to the fact that *the flux-linkage $\phi$ of the magnetic circuit is represented by the actual flux-linkage $Li$ existing in the inductor of the equivalent circuit.*

It is intended to set up a new type of equivalent circuit which corresponds to a physical model of the *magnetic* field underlying the given system. In this new circuit the current represents the flux, and a voltage drop is equivalent to the mmf producing the flux, whereas the resistance represents the reluctance of the magnetic field to the passage of the flux. *The current path in the equivalent circuit will be a model of the flux path in the rotating machine.*

For that purpose let the above equation be divided by $r$:

$$\frac{e}{r} = i + \frac{p}{r}\phi \qquad 2.2$$

Assuming again each term to be represented by a voltage drop, the resultant circuit is shown in Fig. 2.1*c*. This circuit may be called the "dual" of the previous one for the following reasons:

1. The mmf $i$ is represented by a voltage in the second circuit (*c*) and by a current in the first (*b*).
2. The reluctance of the magnetic circuit is represented by a resistor in the second circuit and by an inductor in the first.
3. The resistance of the coil is represented in the second circuit by an inductor, in the first by a resistor.

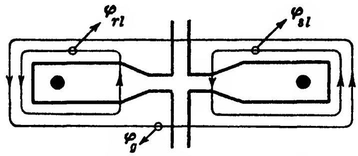
It should be remarked that this dualism existing between the two types of equivalent circuits (one representing a model of the flow of *electric currents*, the other that of the flow of *magnetic fluxes*) is not of the type usually associated with conventional electric circuits.

## MAGNETIC CIRCUIT OF A TWO-WINDING TRANSFORMER

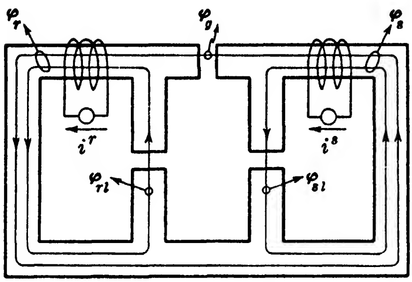
Let one stator and one rotor winding be assumed along the direct axis of the stationary primitive machine (Fig. 2.2*a*). Let also only *one slot* per pole be assumed to exist. When the two currents flow in the same direction at any one instant, the *component* flux lines produced by each current separately are shown in Fig. 2.2*b*. Just as with a two-winding transformer, three different flux lines may be distinguished:

1. The airgap flux $\phi_g$ linking both windings.
2. The stator leakage flux $\phi_{ls}$ linking only the stator winding.
3. The rotor leakage flux $\phi_{lr}$ linking only the rotor winding.

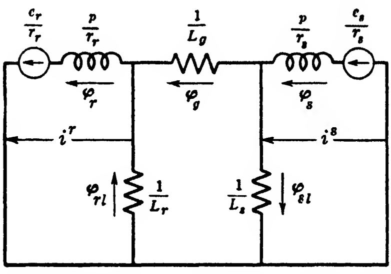

(a) One stator and one rotor winding



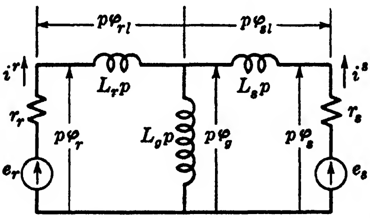

(b) Flux distribution in the slots




(c) The magnetic circuit




(d) Electric circuit representation of flux paths




(e) Conventional "dual" representation of flux paths

FIG. 2.2. Development of the equivalent circuit of a two-winding transformer.

The paths of these three types of flux lines and the reluctances encountered by them are shown again schematically in Fig. 2.2*c*. The reluctance to each type of flux lines is shown by an airgap, and the mmf's producing the flux lines are shown by coils.

## EQUIVALENT CIRCUIT OF THE MAGNETIC PATHS

It is possible to devise an electric circuit to represent the flow of flux lines in a manner analogous to that illustrated in Fig. 2.1*c*. That circuit is shown in Fig. 2.2*d*. In both circuits the following may be noted:

1. The flux lines, $\phi$, are represented as currents.
2. The mmf's, $i$, are represented as voltages.
3. The reluctances, $1/L$, are represented as resistances.
4. The presence of coils, $r$, in the magnetic circuit is represented as inductors.

*The electrical meshes in Fig. 2.2d have the same geometrical configuration as the schematic magnetic paths in Fig. 2.2c.* The voltage equations analogous to Eq. 2.2 are given by the sum of the voltage drops across the inductors and voltage sources.

Strictly speaking, only the configuration of inductors and the mmf's across them represents the magnetic field. The configuration of resistors and impressed voltages is a model of the windings (electric field acting upon the magnetic field).

## THE DUAL EQUIVALENT CIRCUIT

For conventional representation, the previous equivalent circuit is going to be replaced by another type of circuit in which the mmf's are represented not by voltages but by currents. Such a circuit is the "dual" of the former and has the following properties (Fig. 2.2*e*):

1. Resistances, $1/L$, in the former become reactances, $Lp$, in the latter.
2. Reactances, $p/r$, in the former become resistances, $r$, in the latter.
3. Voltage drops, $i$, across the resistors in the former become currents flowing in the latter.
4. Currents, $\phi$, flowing in the former become fluxes in the latter.

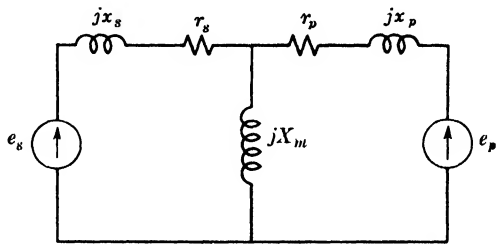
The configuration of the original magnetic circuit differs from the dual in the following way:

5. Impedances in parallel (such as $1/L_s$ and $p/r_s$) become impedances in series ($L_s p$ and $r_s$).
6. Impedances in series (such as the loop of $1/L_s$, $1/L_r$, and $1/L_g$) become impedances in parallel ($L_s p$, $L_r p$, and $L_g p$).

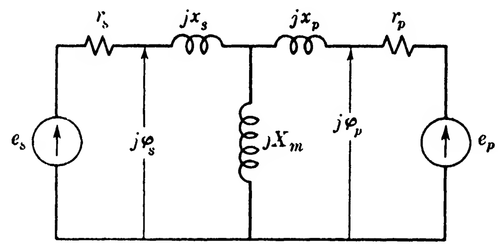
The given dual circuit is the conventional *transient* equivalent circuit of a two-winding transformer valid for both transient and sinusoidal phenomena.

## RESULTANT FLUX LINKAGES

Attention should be called to the fact that the order of arrangement of all resistances *outside* the inductances is not arbitrary but is dictated by physical considerations. The subdivision of the flux lines into *three* components, namely, into an airgap (mutual) and two leakage fluxes —as it has been done hitherto—is an arbitrary design procedure. Actual




(*a*) Artificial model




(*b*) Physical model

FIG. 2.3. Equivalent circuits of a transformer.

measurements on the windings can differentiate, however, only *two* flux-components:

1. Resultant flux linking the stator winding.
2. Resultant flux linking the rotor winding.

The rates of change of these *resultant* flux linkages $\phi_s$ and $\phi_r$ can actually be measured only as voltages across the reactances. Figure 2.2*e* shows both types of subdivisions.

On the other hand, if the equivalent circuit of a transformer (or induction motor) is represented as in Fig. 2.3*a*, where the resistances separate the leakage and mutual inductances (as many textbooks show), then *the concept of "resultant flux" linking a winding is absent.* In rotating

machinery the introduction of such a concept is important for at least three purposes:

1. The calculation of torque on the rotor.
2. The calculation of generated voltages during small oscillations of the rotor.
3. Sudden-short-circuit studies when constant flux linkages are assumed.

## MAGNETIC PATHS OF A FOUR-WINDING TRANSFORMER

Let it be assumed next that two stator and two rotor windings exist along one of the axes of the machine (Fig. 2.4*a*). The flux distribution is shown in Fig. 2.4*b* and *c*. It should be noted that:

1. The airgap flux (mutual flux) surrounds all four windings.
2. Mutual fluxes $\phi_{12}$ and $\phi_{34}$ exist between two windings that lie on the same magnetic structure. They are the so-called "slot-leakage" fluxes.
3. Each of the windings has flux lines (partly slot leakages and partly end leakages) that link only that particular winding.
4. In the first and last windings $\phi_l$ and $\phi_s$ can be combined into one flux, if no further windings exist on the stator and rotor. They have not been combined here in order to allow the introduction of more windings if needed.

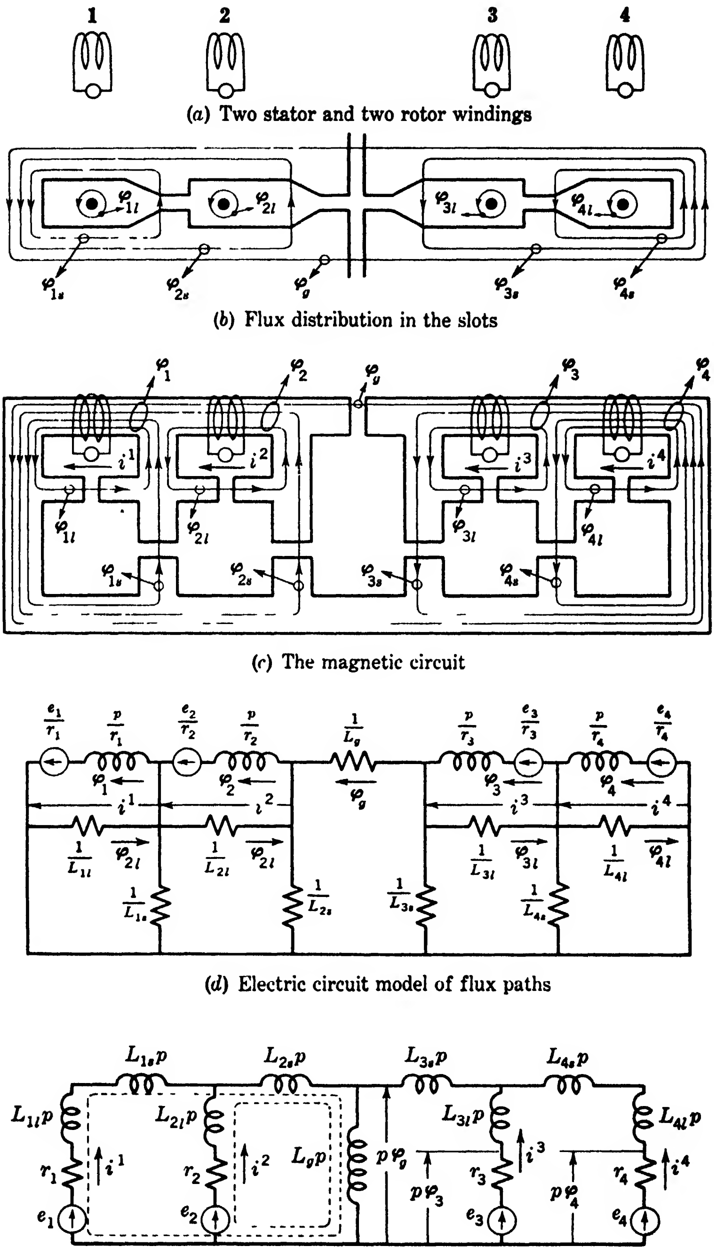
The electrical equivalent of the magnetic flux paths is shown in Fig. 2.4*d*, having the same configuration as Fig. 2.4*c*. Again each reluctance is represented by a resistance. It should also be noted that the current representing the "resultant" flux linkages of each winding ($\phi_1$ to $\phi_4$) is assumed to pass through an inductor.

It is, of course, obvious that the magnetic circuit shown is not the most general possible for a four-winding transformer. For instance, here no flux lines exist that are mutual exclusively to windings 2 and 4, or to windings 2, 3, and 4, etc. This absence of several mutual fluxes is the reason for the simplicity of the dual circuit to be shown presently.

## TRANSIENT EQUIVALENT CIRCUIT OF A FOUR-WINDING TRANSFORMER

Interchanging resistances and inductances, also series and parallel branches, the final "dual" magnetic equivalent circuit is shown in Fig. 2.4*e*. The circuit represents the transient performance of the four windings. The following observations may be made:

1. The currents flowing in each winding are represented by the currents in the *vertical* branches in which the resistances and impressed voltages are placed.




(a) Two stator and two rotor windings

(b) Flux distribution in the slots

(c) The magnetic circuit

(d) Electric circuit model of flux paths

(e) Conventional "dual" representation of flux paths

FIG. 2.4. Development of the equivalent circuit for *one* axis.

2. Each winding is represented in the equivalent circuit by a mesh that passes through a vertical branch with a resistance and also passes through the vertical airgap inductance $L_g p$, as shown by dotted lines. The airgap inductance is mutual to all meshes, the horizontal inductances (the slot leakages) are mutual to neighboring windings, whereas the vertical inductances (the end leakages) are not mutual with any of the other meshes. Similarly, the resistances and impressed voltages are not mutual with any other winding.

Hence, whereas in the rotating machine each winding is electrically isolated, in the equivalent circuit the windings are electrically coupled. The existence of this coupling demands the assumption that the reactances of the primitive machine must be so defined that *each winding has the same number of turns.* Later on in the study of commutator machines, when the necessity will arise to introduce a different number of turns for some of the windings, the corresponding meshes will have to be electrically isolated.

3. The time rate of change of the resultant flux linking each winding is given by the voltage measured across all the inductors of each mesh, as shown. Hence the existence of this concept forces the arrangement of the resistors outside all inductors (next to the impressed voltages).

4. All impressed voltages appear connected to a common ground, which is not the case in an actual machine. This fact, however, does not cause any difficulties.

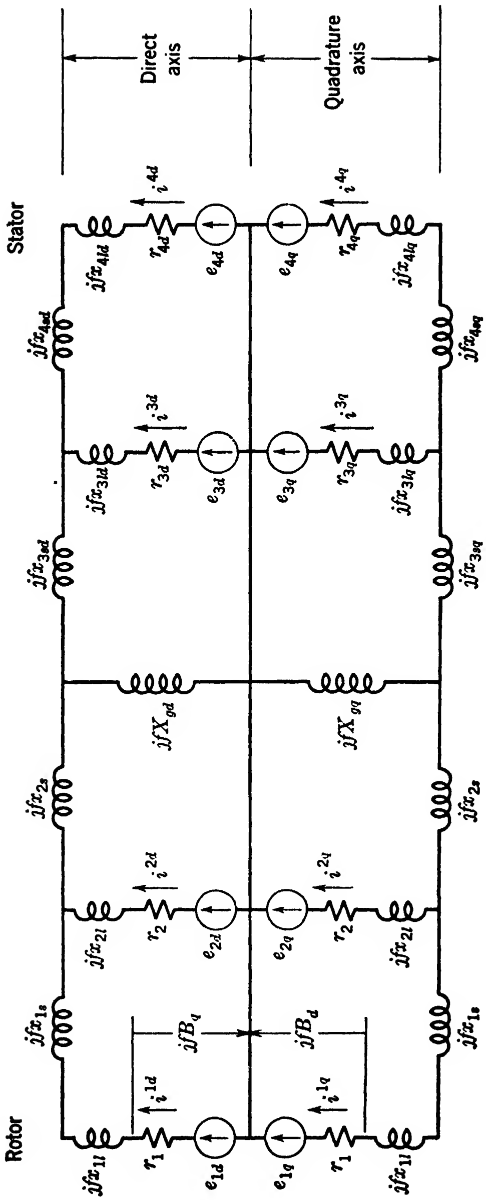
5. The *horizontal* currents represent the resultant magnetomotive forces ($H$) acting between the various layers of windings and producing slot-leakage fluxes. The currents in the horizontal branch next to the airgap represent the sum of the stator (or rotor) currents.

6. The flux linkages ($\phi = Li$) in each inductor of the equivalent circuit are equal to the corresponding flux linkages in the actual machine.

## THE PRIMITIVE MACHINE AT STANDSTILL

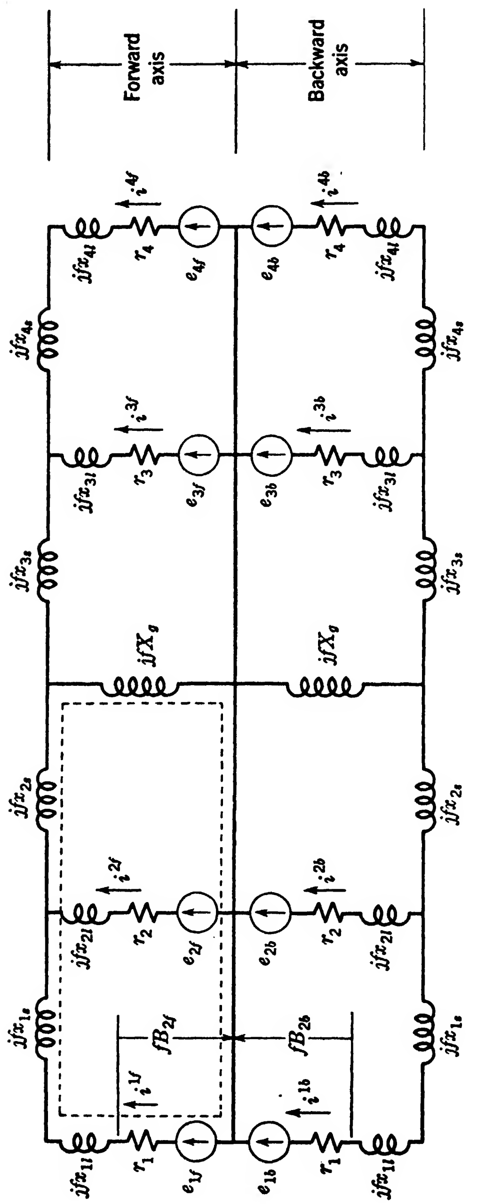
If two sets of four windings are assumed at right angles in space, all stationary, their magnetic flux lines are independent of each other. Hence *their equivalent circuits are also electrically isolated with no mutual couplings between them,* as shown in Fig. 2.5$a$. (The direct-axis quantities have a subscript, $d$, the quadrature-axis quantities have $q$.) If the inductances are left in the form $Lp$, the circuits hitherto developed are valid for *transient* operation, assuming the rotor at standstill.

Here the simplified assumption will be made that all currents and voltages in each mesh of the machine at rest are *sinusoidal in time,* that all are of the same frequency $f$, and that in each coil of the equivalent circuit only *one* sinusoidal current flows. With such assumptions *every*

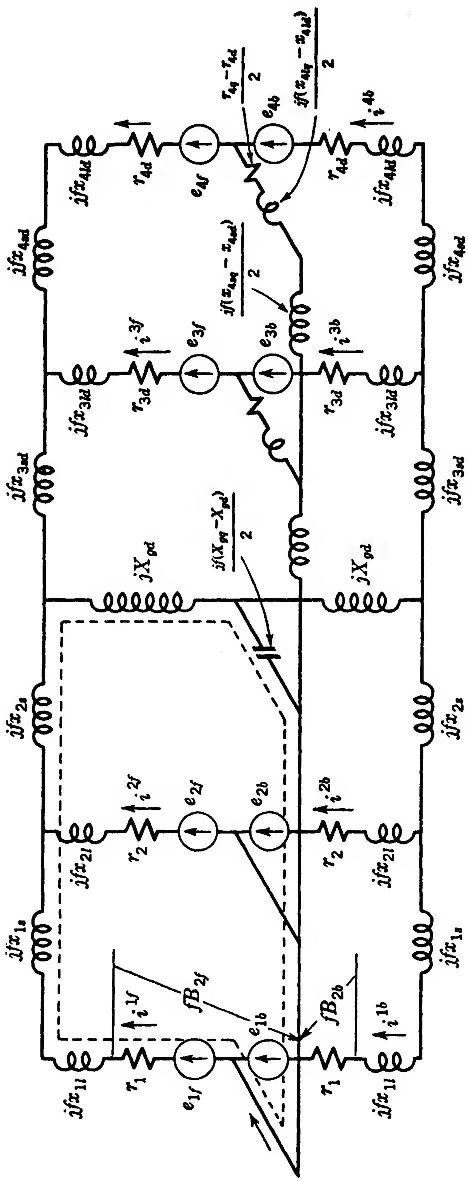



*(a)* Direct and quadrature axes

FIG. 2.5. Development of the equivalent circuit for two axes.




(b) Forward and backward axes, balanced windings




(c) Forward and backward axes, unbalanced windings

FIG. 2.5. *Continued.*

*$p$ is replaced by $jf\omega$*, so that $pL = jf\omega L = jfx$. The frequency $k$ in $\omega = 2\pi k$, at which the circuit reactances $x = \omega L = 2\pi kL$ are calculated (usually 60 cycles), will be denoted in the following by unity, so that the impressed frequency $f$ is expressed as a fraction of unity. In general, *the impressed frequency $f$ will be assumed to be a variable quantity.*

## SYMMETRICAL COMPONENTS

When $p$ is replaced by $jf\omega$, all waves in the machine become sinusoidal in time. It is possible to replace the stationary alternating waves along the **d** and **q** axes with rotating waves of *constant magnitude* by introducing two hypothetical "sequence" axes on each layer of winding, one rotating forward in the same direction as the rotor (**f** axis), the other rotating backward (**b** axis). The angular speed of rotation of these waves is $f\omega$ ($p = jf\omega$). The forward **f** direction is assumed from **d** to **q**.

The advantage of the introduction of sequence axes is that by their use the alternating waves will be replaced by rotating waves with constant magnitude. Hence in the analysis of a rotating machine the distinction between induced and generated voltages will disappear *and all voltages will become generated voltages.* The speeds to consider then will be as follows:

1. The speed of the fluxes, $f\omega$.
2. The speed of the rotor $p\theta = v\omega$.
3. The relative speeds between flux and rotor $(f + v)\omega$ and $(f - v)\omega$.

More detailed and complete explanation will be presented in the next chapter.

## TRANSFORMATION TO SEQUENCE AXES

As a simple example illustrating the derivation of the sequence circuit of Fig. 2.5*b*, let first a two-mesh network be given with *no* mutual *impedance* between them (Fig. 2.6*a*). Its equations are

$$e_d = z_d i^d \qquad \text{and} \qquad e_q = z_q i^q \tag{2.3}$$

Let symmetrical components be introduced by

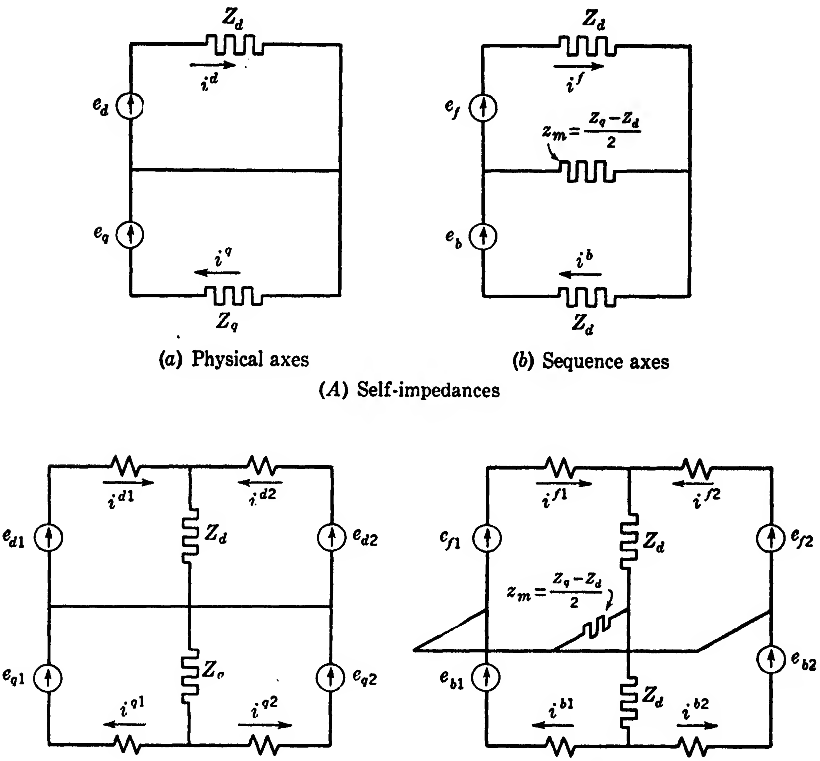
$$\begin{array}{l|l} i^d = (i^f + i^b)/\sqrt{2} & e_d = (e_f + e_b)/\sqrt{2} \\ i^q = -j(i^f - i^b)/\sqrt{2} & e_q = -j(e_f - e_b)/\sqrt{2} \end{array} \tag{2.4}$$

The new equations are

$$\begin{aligned} e_f &= \frac{z_d + z_q}{2} i^f + \frac{z_d - z_q}{2} i^b = (z_d + z_m) i^f - z_m i^b \\ e_b &= \frac{z_d - z_q}{2} i^f + \frac{z_d + z_q}{2} i^b = -z_m i^f + (z_d + z_m) i^b \end{aligned} \tag{2.5}$$

where $z_m = (z_q - z_d)/2$.

The new circuit is shown in Fig. 2.6*b*. The impedance $z_d$ remains an impedance in both branches, whereas the difference between the two self-impedances, $z_m$, becomes a mutual impedance. If the two original impedances are equal ($Z_q = Z_d$), there is no mutual coupling between



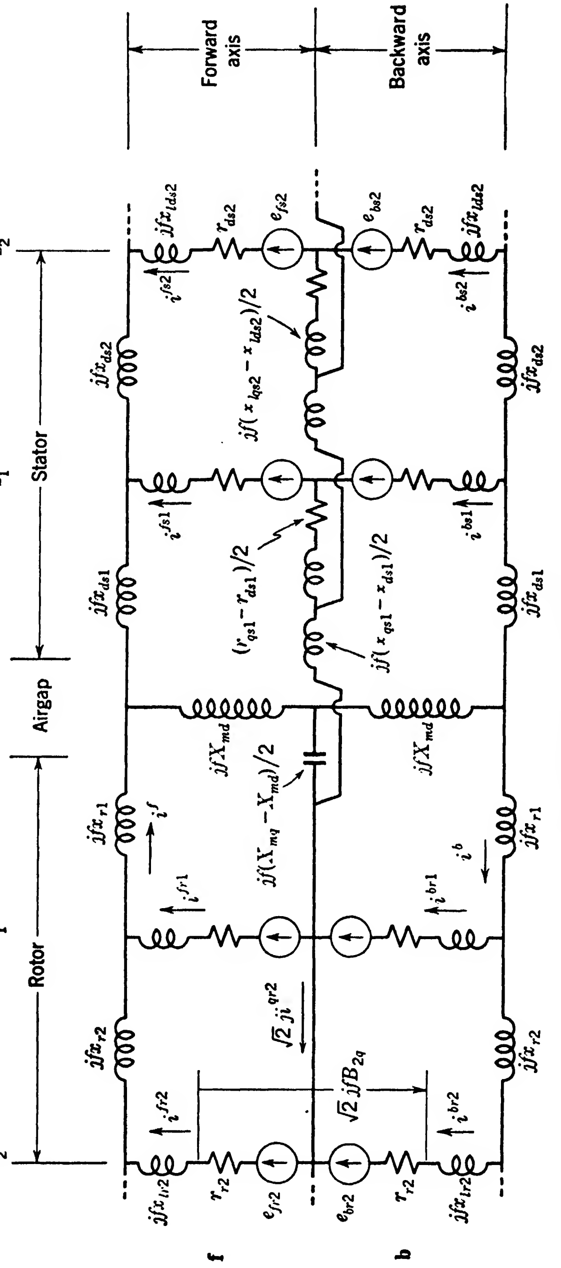

(*a*) Physical axes (*b*) Sequence axes
(*A*) Self-impedances

(*c*) Physical axes (*d*) Sequence axes
(*B*) Mutual impedances

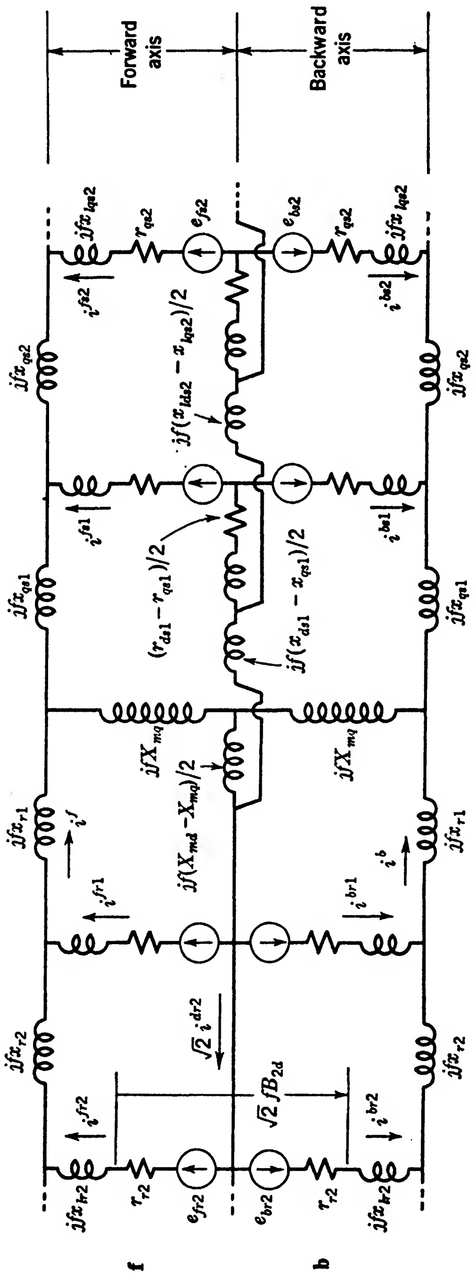
FIG. 2.6. Transformation of two-phase networks to sequence axes.

the sequence meshes, and the sequence network is the same as the original network. However, the impressed voltages and currents in the former are sequence quantities.

Let along each axis a two-mesh network exist with a mutual impedance $z_d$ or $z_q$ between each two meshes (Fig. 2.6*c*). Again no impedance exists between the **d** and **q** meshes. If sequence axes are introduced, the difference between $z_d$ and $z_q$ again becomes a mutual impedance between the **d** and **q** meshes, as shown in Fig. 2.6*d*. Now $z_m$ is mutual between *four* meshes and lies in a branch extending in the third dimension.




(a) Using direct-axis constants




(b) Using quadrature-axis constants

FIG. 2.7. The primitive induction machine at standstill (revolving-field theory).

## EQUIVALENT CIRCUITS ALONG THE SEQUENCE AXES

If, now, the winding constants on the quadrature axis are identical with those on the direct axis (no saliency exists and the windings on each layer are uniform), then along each of the new axes (**f** and **b**) the equivalent circuits are identical with those along the **d** axis (Fig. 2.5*b*). All constants along both **f** and **b** axes carry the same subscript, $d$ (or no subscript at all). Again no mutual reactances appear between the **f** and **b** axes, and the two sets of four meshes are electrically isolated. The voltages impressed now represent forward- and backward-rotating impressed voltages ($e_f$ and $e_b$), and the currents represent $i^f$ and $i^b$.

If a quadrature-axis constant, $x_q$, differs from that of the direct axis, $x_d$, then *the difference* $(x_q - x_d)/2$ *appears as a mutual impedance between the two sequence axes* $f$ *and* $b$. These mutual impedances are arranged in the form of four meshes in exactly the same manner as the **f** and **b** impedances are. The plane of these meshes is tilted at right angles to the plane of the paper on Fig. 2.5*c* for easier visualization. The impedances of the original **f** and **b** meshes all carry the direct-axis subscripts.

Each winding of the machine is represented again by a mesh (shown by dotted lines) passing through a vertical resistance $r$, the vertical airgap reactance $X_g$ and the horizontal reactance $(X_{gq} - X_{gd})/2$ representing the saliency of the machine. *The saliency reactance is common to all eight meshes.*

The new equivalent circuit along the sequence axes is shown in Fig. 2.7*a* with the mutual branches flattened out in the plane of the paper. The nomenclature has been changed to distinguish the stator ($s$) and rotor ($r$) windings. Also the windings next to the airgap have the subscript 1, whereas the windings in the next layer have the subscript 2.

The equivalent circuit along the sequence axes is far more complicated than the one along the physical **d** and **q** axes (Fig. 2.5*a*). This apparent complication will pay dividends, though, when rotation of the rotor conductors is considered.

## ANOTHER FORM OF THE SEQUENCE NETWORK

In the sequence equivalent circuits given, the direction of the currents is assumed to be *clockwise in all rotor meshes*. With this convention the meshes contain three times as many direct-axis quantities, in general, as quadrature axis quantities.

In the synchronous machine studies, occasions will arise in which the resultant circuits may be simplified by assuming another form of the primitive sequence network in which the direction of the flow of the forward rotor currents $i^{fr}$ in the rotor meshes is clockwise but those of

the backward rotor currents is counterclockwise (Fig. 2.7*b*). As a result, *the position of the direct and quadrature axis quantities is simply interchanged. Hence in the main branches occurs* $x_q$ *and in the common branches* $(x_d - x_q)/2$, *etc.* The direction of the **b** voltages is also changed. Now the meshes contain, in general, three times as many quadrature-axis quantities as direct-axis quantities.

## THE APPEARANCE OF FLUX DENSITIES B

It was mentioned that the flux-density wave **B** in the smooth member is equal to the flux-linkage wave $\boldsymbol{\phi}$ and lies at right angles to it in space

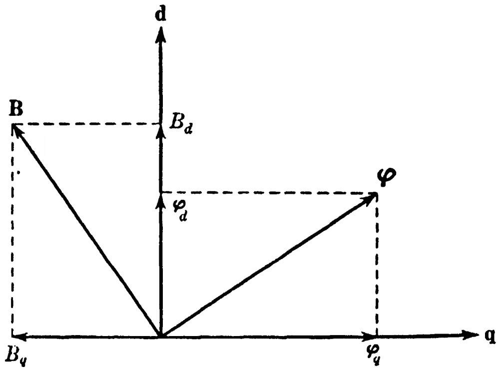



FIG. 2.8. Relations between $\boldsymbol{\phi}$ and **B**.

(Fig. 2.8). Assuming **d** and **q** axes, the relation between them is (for the primitive *induction* machine under analysis)

$$B_d = \phi_q \qquad \text{and} \qquad B_q = -\phi_d \tag{2.6}$$

However, along symmetrical components the relation is quite different. For instance,

$$\phi_f = \frac{1}{\sqrt{2}}(\phi_d + j\phi_q) = \frac{1}{\sqrt{2}}(-B_q + jB_d) = \frac{1}{\sqrt{2}}j(B_d + jB_q) = jB_f$$

Hence for the primitive induction machine the relation between $\boldsymbol{\phi}$ and **B** is

$$B_f = -j\phi_f \qquad \text{and} \qquad B_b = j\phi_b \tag{2.7}$$

In the sequence network the voltages induced across the inductors also represent the resultant flux linkages of the windings $p\phi_f = jf\phi_f$ and $p\phi_b = jf\phi_b$. But now the $j\phi_f$ may be replaced by $-B_f$ and $j\phi_b$ by $B_b$, which do not contain $j$.

Hence *in the sequence networks the voltages measured across the inductances represent outright the flux densities* **B** without any $j$ factor. This is in line with the facts that along the sequence axes only generated voltages appear and they are directly proportional to $B$.

## PHYSICAL QUANTITIES IN THE SEQUENCE CIRCUITS

It is interesting that some of the currents and flux densities that ordinarily occur in the **d** and **q** equivalent circuits may be found also in the sequence networks. In particular:

1. The difference in the sequence currents $i^f - i^b$ flowing in the common branches of Fig. 2.7*a* represents $\sqrt{2}ji^q$ by definition.

2. The difference in the sequence flux densities $B_f - B_b$, measured as the differences of potential across two sequence meshes, represents $\sqrt{2}jB_q$.

In the second form of the primitive sequence networks of Fig. 2.7*b* (in which the signs of $i^b$ and $B_b$ have been reversed) the current in the common branches represents $\sqrt{2}i^d$, and the differences of potential across the two meshes give $\sqrt{2}fB_d$.

The reverse, however, is not true. In the physical primitive networks of Fig. 2.5*a* representing $i^d$ and $i^q$ there is no place for $i^f$ or $i^b$, whose definitions require the existence of $ji^q$. However, later on another form of the physical network will be developed that does contain $ji^q$ in place of $i^q$ and hence contains also some of the sequence quantities.

# 3 THE PRIMITIVE MACHINE AT CONSTANT SPEED

## OUTLINE OF THE STEPS

Whereas the equivalent circuits for a *stationary* primitive machine were established first for the physical **d** and **q** axes and afterwards only for the sequence axes, the steps will be reversed when the primitive machine rotates. For rotation it will be easier to establish the equivalent circuit along the sequence axes first. This step will require only a knowledge of the actual frequencies of currents existing in the conductors (the "absolute" frequencies). The reverse transformation from the sequence axes to the physical axes will follow in an automatic manner.

Two types of equivalent circuits will be constructed along the physical axes. One will contain the same currents as the actual machine; the other will contain $ji^q$ instead of $i^q$. Although the correlation of this last network with the actual machine is not so clear-cut (because of the appearance of $ji^q$), the network itself has a simpler form. Both types of primitive networks (along the physical axes) will be used in the study of specific rotating machines.

All developments will be based upon the primitive *induction* machine, in which the physical reference frames are stationary in space. The step to the other primitive machine, with rotating reference frames—to be used for synchronous machines—will involve merely a change in the sign of the rotor velocity $p\theta = v$.

*The direction of the physical motion of the rotor $v$ will always be from* **d** *to* **q** (the same as the "forward" direction of the fluxes).

## THE "ABSOLUTE" FREQUENCIES

It should be recalled that in the presence of sequence axes **f** and **b** no induced voltages exist. All fluxes rotate with a constant magnitude, and all voltages due to them are generated voltages. Attention will now be directed to the determination of the frequencies of these generated voltages.

When the rotor conductors are stationary, the flux-density wave along the **f** and **b** axis of any rotor winding generates fundamental or "base" frequency $f$ voltages in the stationary rotor conductors. The frequency of the currents in the conductors is also $f$ ($f$ is a variable quantity). If, however, the rotor conductors rotate clockwise with an angular velocity $p\theta = v\omega$, then the flux-density wave along the **f** axis generates a slip-frequency $f - v$ voltage in the rotor conductors. On the other hand, the backward rotating flux-density wave along the **b** axis generates in the same rotor conductors a voltage of $f + v$ frequency.

To summarize, when the rotor rotates with an angular velocity $v\omega$, the frequency of all quantities in the conductors of the various windings is as follows:

1. In all stator **f** and **b** axes it is $f$.
2. In all rotor **f** axes it is $f - v$.
3. In all rotor **b** axes it is $f + v$.

The frequencies of currents (and voltages), $n$, in the actual *conductors* will be called "absolute" frequencies, to distinguish them from the "base" frequencies, $f$, existing along the *physical* reference axes **d** and **q**. The latter frequencies are $f$ in both stator and rotor, irrespective of whether the rotor rotates.

## OTHER DEFINITIONS OF "ABSOLUTE" FREQUENCIES

The "absolute" frequencies may also be looked upon as the frequencies appearing along a set of reference frames that is rigidly connected to both stator and rotor conductors. This set of reference frames will be introduced only in Chapter 11 when time harmonics are considered.

It should be noted that the absolute frequencies are also identical with the *slip frequency of the reference axes* **f** and **b** *with respect to the conductors,* since the speed of the fluxes and the reference axes is the same.

The definition of the "absolute" frequencies as actual current and flux frequencies in the conductors is preferred here, as it will facilitate the calculation of torques of various frequencies.

## VOLTAGE EQUATIONS IN THE MACHINE

It is easy to establish the voltage equations along the sequence axes since the *voltage due to the presence of fluxes all are generated voltages* and the *magnitude* of these generated voltages depends only upon the relative speed of the sequence axes **f** and **b** and the rotor conductors. *This relative speed is given by the "absolute" frequencies.* Hence the equation of voltage along a rotor **f** axis, for instance, has the form

$$e = ir + j(f - v)\Sigma ix \qquad 3.1$$

since the absolute frequency associated with the rotor forward axis is $f - v$. In this equation $x$ represents the various self-inductances and mutual inductances of the stationary windings, calculated at unit frequency, and the term $j\Sigma ix$ represents the resultant flux, due to all stator and rotor currents, linking the rotor **f** winding.

For each winding a similar equation may be established in which the magnitude of the *generated* voltage is proportional to its "absolute" frequency. These equations refer to the sequence axes attached to the *actual machine* in which all windings are isolated from each other.

However, in the equivalent circuit the meshes (representing the windings) are mutually coupled by reactances, and these equations compel each coupling reactance to assume several values simultaneously, for instance, $(f - v)x$, $(f + v)x$, and $fx$. Therefore, an equivalent circuit cannot be established when $v$ is different from zero.

## VOLTAGE EQUATIONS IN THE EQUIVALENT CIRCUIT

However, it should be observed that in the equivalent circuits of the stationary primitive machine of Fig. 2.7 the impressed voltages, $e$, and resistances, $r$, either are isolated for each mesh (in the vertical branches) or are coupled (on the stator only) with such meshes that have identical absolute frequencies. This observation suggests that, if it is assumed that in all stator and rotor meshes of the equivalent circuit the frequencies are identical and all are unity, then the mutual couplings will also be identical, while the $r$'s and $e$'s will differ.

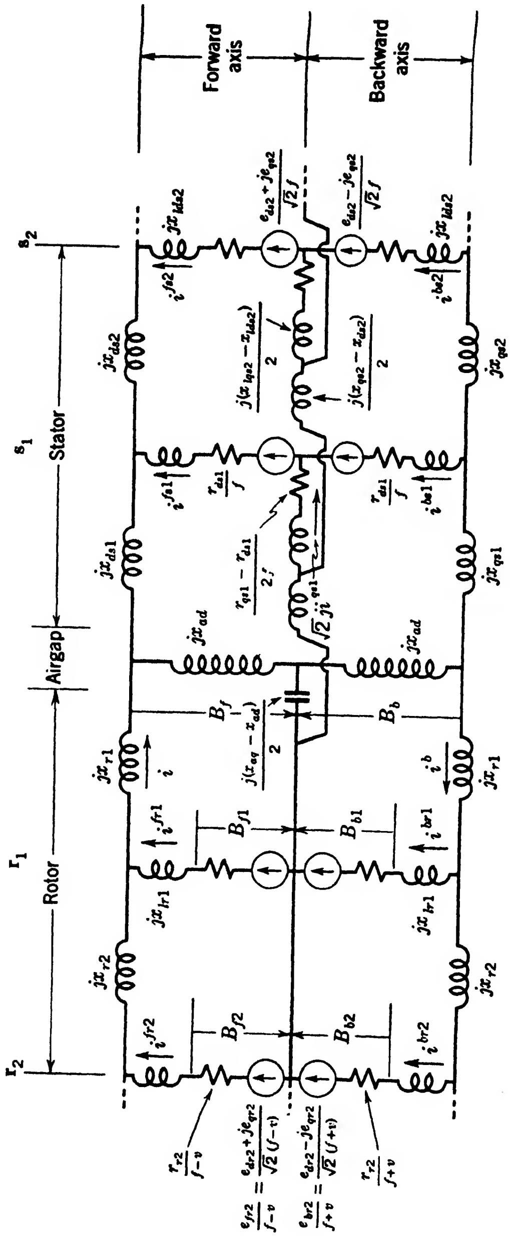
Hence, if the voltage equation of each winding of the machine along the sequence axes is divided by its absolute frequency, the equations may apply also for the equivalent circuit of the stationary primitive machine (Fig. 2.7). The equation of any rotor **f** mesh becomes, then,

$$\frac{e}{f - v} = i\frac{r}{f - v} + j\Sigma ix \qquad 3.2$$

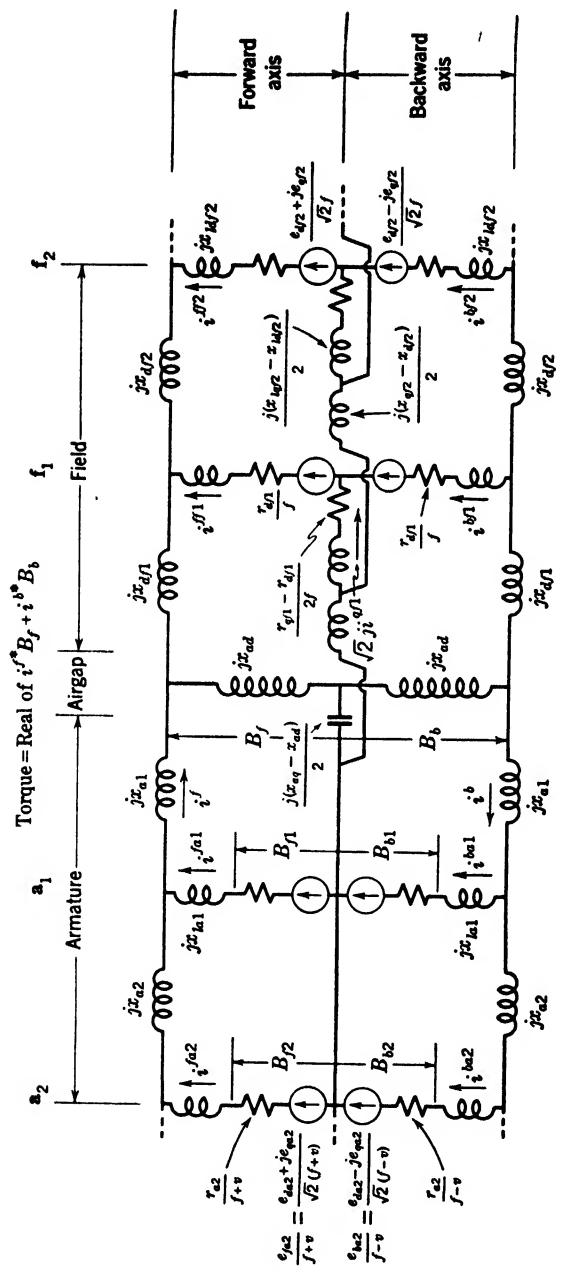
and the coefficients of all $x$ become the same, namely, unity.

*Hence, if all resistances and impressed voltages of Fig. 2.7 are divided by their respective absolute frequencies, the same stationary network—shown again in Fig. 3.1a—represents the performance of the primitive rotating machine excited along any or all axes by a single variable frequency $f$.*

In the transition from a stationary machine to a rotating machine, the configuration of the model remained undisturbed. The magnitude of all inductors—representing the magnetic field—also remained unchanged. The only modification occurred in the magnitude of the resistors and impressed voltages, representing the electrical field.




(a) Induction machine




(b) Synchronous machine

FIG. 3.1. The primitive machine rotating (revolving-field theory).

## THE PRIMITIVE SYNCHRONOUS MACHINE

The same theory holds irrespective of whether the physical reference frame (**d** and **q**) is attached to the stator (Fig. 3.1*a*) or to the rotor (Fig. 3.1*b*) with the difference that:

1. The velocity $v$ assumes an opposite sign when the stator rotates instead of the rotor.
2. The flux densities $B$ also assume opposite signs.

The subscripts in the new networks are given synchronous-machine terminology ($a$ for armature, $f$ for field).

To avoid lengthy circumlocution, the equivalent circuits along the **d** and **q** axes will also be called "cross-field" circuits, and those along the **f** and **b** axes will also be called "revolving-field" circuits, irrespective of whether the circuits refer to synchronous or induction machines.

## THE PRIMITIVE POLYPHASE MACHINE

Let it be assumed that in a polyphase machine:

1. Both magnetic structures are smooth (no saliency).
2. All layers of windings are balanced.

If this is the case no mutual coupling exists between the forward and backward meshes and each primitive network splits into two isolated networks. Let it be assumed, in addition, that:

3. Only positive-sequence (**f**) voltages are impressed.

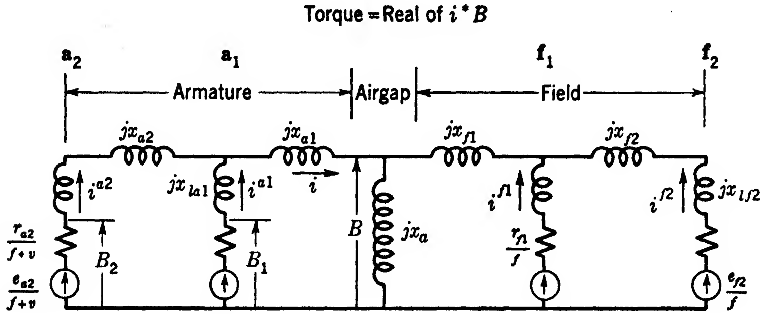
In this case the primitive equivalent circuit consists of only the forward meshes, as shown in Fig. 3.2. The expression "primitive *balanced* machine with *balanced* voltages" will be shortened to "primitive polyphase machine."

## PARALLELISM BETWEEN MACHINES AND CIRCUITS

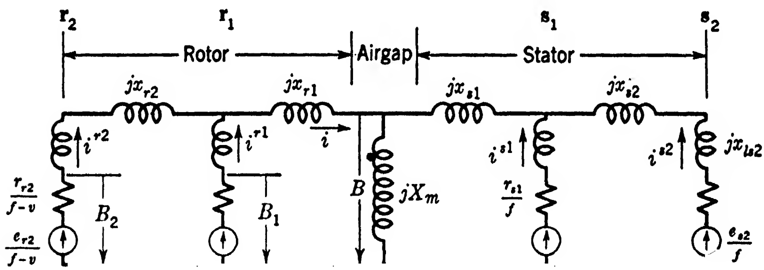
It is important to note that, whereas the currents in the equivalent circuits are identical with the currents in the actual machine (along the sequence axes), that is not the case for the voltages. *Voltages in the revolving-field equivalent circuits correspond to flux densities in the actual machines* (expressed as the ratio [voltage]/[frequency] = flux-density). This fact is, of course, due to the division of the machine voltage equations by the absolute frequencies.

If the machine torque (as defined by Maxwell) is expressed as the scalar product of a flux density and a current $T = iB$, *then a torque in the machine must be represented by a power = current × voltage in the equivalent circuit.* This statement refers not only to a constant steady-state torque but also to an alternating torque, such as the damping or synchronizing torque of an oscillating machine. The constant (d-c) torque

is defined by a *real* power, whereas an oscillating torque is defined by both real and reactive powers (as will be shown in Chapter 4).




(*a*) Polyphase synchronous machine




(*b*) Polyphase induction machine

Fig. 3.2. Primitive polyphase machine (revolving-field theory).

## CONSTANT TORQUE CALCULATIONS

The instantaneous torque $T$ on each rotor winding may be found by Maxwell's force equation ($T = \mathbf{iB}$) as the scalar product of the resultant rotor flux-density wave **B** and the resultant rotor current wave **i**. When $B$ and $i$ are both complex numbers, then for each layer of winding the *constant* (d-c) torque is

$$T = \text{Real of } i^{fr*}B_{fr} + i^{br*}B_{br} \qquad 3.3$$

represented by a *real* power passing through each rotor mesh of the equivalent circuit. (An asterisk represents the conjugate of a complex number.)

Since the torques due to the rotor leakage fluxes cancel, it is sufficient to use the airgap fluxes $B_g$ in the torque calculations, instead of the resultant fluxes $B_r$.

In the event of several layers of windings on the rotor the individual currents may be added. Their resultant current flows in the horizontal coil next to the airgap, as shown in Fig. 3.1. Hence, in the above formula for constant torque, $i^{fr}$ and $i^{br}$ are the resultant rotor currents, whereas $B_{fr}$ and $B_{br}$ may be the voltages across the airgap reactances, as shown.

The above torque formula is equivalent also to the statement that the torque is given by the rotor losses in the **f** meshes minus the rotor losses in the **b** meshes. The torques thus found are always expressed in "synchronous watts." The total torque is changed from synchronous watts to pound-feet by

$$\text{Pound-feet} = \frac{\text{Synchronous watts} \times 33{,}000 \times (\text{Number of poles})}{2\pi(2 \times 60 \times f)746} \tag{3.4}$$

## RETURN TO THE PHYSICAL AXES

Now that the performance of the primitive rotating machine has been expressed in the form of a stationary network in one type of reference frame (namely, in the sequence axes **f** and **b**), it should be comparatively easy to establish stationary networks in other types of reference frames, particularly in the original **d** and **q** frame.

In transforming back to the original **d** and **q** axes, the stator mesh quantities assume their former values in Fig. 2.5 with the difference that all resistances and impressed voltages are divided by $f$. However, the *rotor mesh quantities cannot assume their previous values*, since in the meantime they were divided by the absolute frequencies, which were different for the **f** and for the **b** meshes, respectively.

Let the equations of any pair of rotor meshes along the sequence axes be written as

$$e_f = Z_f i^f$$

$$e_b = Z_b i^b$$

Let the original variables be reintroduced by

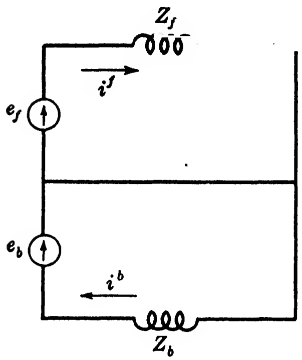
$$\begin{aligned} i^f &= (i^d + ji^q)/\sqrt{2} \qquad & e_f &= (e_d + je_q)/\sqrt{2} \\ i^b &= (i^d + ji^q)/\sqrt{2} \qquad & e_b &= (e_d - je_q)/\sqrt{2} \end{aligned} \tag{3.5}$$

By substitution the following equations are found:

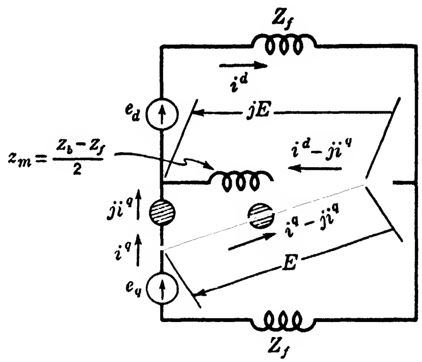
$$e_d = \left(\frac{Z_f + Z_b}{2}\right) i^d + j\left(\frac{Z_f - Z_b}{2}\right) i^q = (Z_f + Z_M)i^d - jZ_M i^q$$
$$e_q = -j\left(\frac{Z_f - Z_b}{2}\right) i^d + \left(\frac{Z_f + Z_b}{2}\right) i^q = jZ_M i^d + (Z_f + Z_M)i^q \qquad 3.6$$

where $Z_M = (Z_b - Z_f)/2$.

Just as in Fig. 2.7, here also $Z_f$ remains in both branches (Fig. 3.3*b*) while the difference between the two self-impedances $Z_M$ becomes again




(*a*) Along sequence axes




(*b*) Along physical axes

FIG. 3.3. Transformation of two meshes.

a mutual impedance. However, this mutual impedance is not reciprocal; that is, $i^q Z_M$ has opposite sign to $i^d Z_M$. Such an expression cannot be represented by a stationary network.

Two courses are open to remedy the situation. Either the equations are transformed and brought to a new form in which the mutuals are reciprocal, or some extra generators (the so-called "phase shifter") are introduced to take care of the non-reciprocal nature of the mutuals. Both of these methods will be used, as each has its particular advantage, to be explained presently.

## PHASE SHIFTERS

The role of a phase shifter (shown in Fig. 3.3*b*) is to rotate in time phase the current and the voltage appearing in a mesh by the same angle $\alpha$. In this present case that angle is 90°. A phase shifter may be represented by a generator *in series* with $Z_f$ and by another generator *in shunt* with $Z_M$. The generator voltage values are automatically

adjusted to inject the required angle of shift for the current from $i$ to $i\epsilon^{j\alpha}$ (in the series generator) and for the voltage from $e$ to $e\epsilon^{j\alpha}$ (in the shunt generator). When $\alpha = 90$, $\epsilon^{j90} = j$.

In writing the voltage equation, for instance, for the mesh containing $e_q$ and the shunt generator,

$$e_q = i^q Z_f - (i^d - ji^q)\left(\frac{Z_b - Z_f}{2}\right)(-j)$$

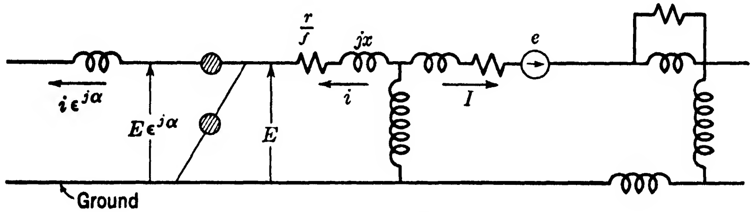
the actual voltage drop across the mutual is multiplied by $-j$ to represent the voltage drop across the shunt generator. Rewriting the equation,

$$\begin{aligned} e_q &= i^q Z_f + i^q\left(\frac{Z_b - Z_f}{2}\right) + ji^d\left(\frac{Z_b - Z_f}{2}\right) \\ &= i^q\left(\frac{Z_b + Z_f}{2}\right) - ji^d\left(\frac{Z_f - Z_b}{2}\right) \end{aligned}$$

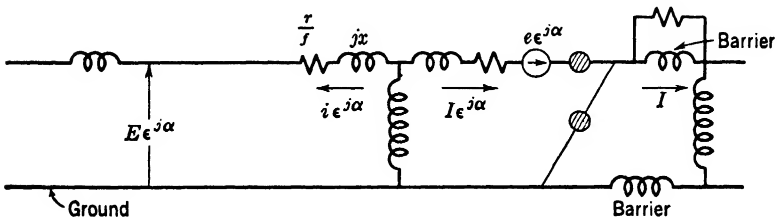
which is the original Eq. 3.6.

## THE SHIFTING OF PHASE SHIFTERS

The two generators forming a phase shifter may be shifted from one part of a mesh into another part of the mesh or into other meshes, provided that certain conditions are satisfied. By writing the equations




(a) Before shifting




(b) After shifting

FIG. 3.4. Shifting a phase shifter.

before and after the shifting, it can be ascertained that one of the conditions is that the end of the shunt generator which is tied to the ground (the common branch) cannot pass through an impedance or a generator. Hence the mutual impedance due to a saliency effect (Fig. 3.1) acts as a barrier in the sequence networks, as shown in Fig. 3.4. Similarly, the mutual rotor resistance in the cross-field networks of Fig. 3.5 acts as a boundary.

Another condition is that the opposite end of the phase shifter which passes through an impedance cannot pass if the impedance is mutually coupled to another mesh, as shown in Fig. 3.4*b*.

As a phase shifter is shifted, all currents and voltages that have been passed over will be shifted by the required angle, as shown in Fig. 3.4*b*.

## CROSS-FIELD EQUIVALENT CIRCUIT OF THE PRIMITIVE MACHINE

If a 90° phase shifter is placed in each rotor axis across the mutual resistances, the two primitive equivalent circuits are as shown in Fig. 3.5. The effect of rotation is twofold:

1. A mutual resistance appears in the rotor axes in conjunction with a phase shifter. Its value is, by Fig. 2.6*b*,

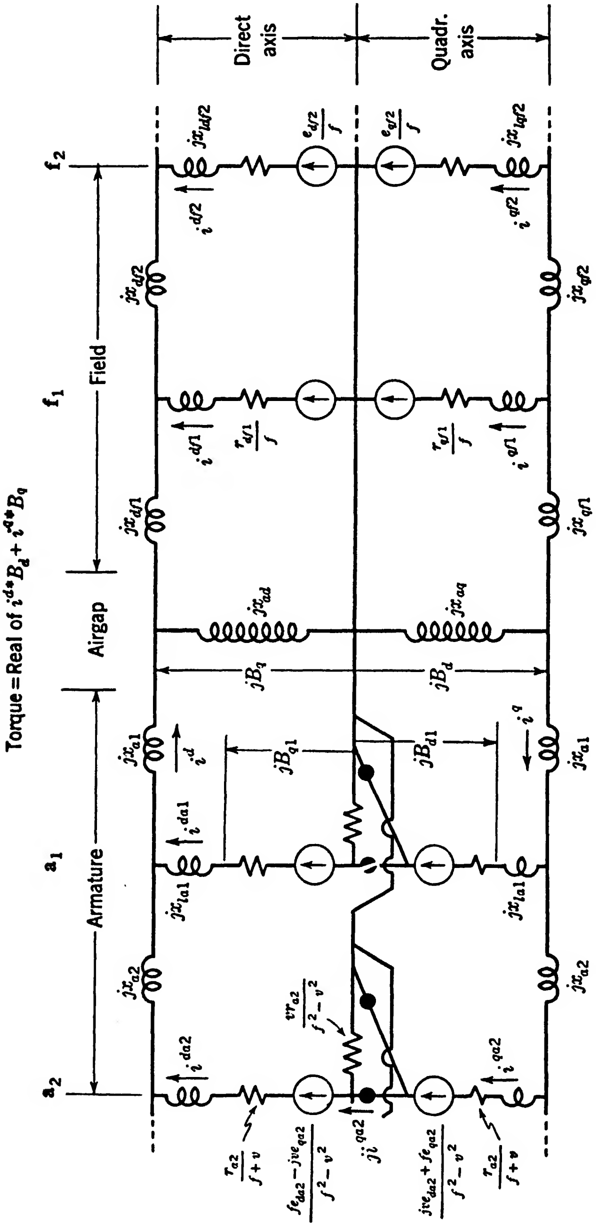
$$\frac{1}{2}\left[\frac{r}{f-v} - \frac{r}{f+v}\right] = \frac{-rv}{f^2 - v^2} \tag{3.7}$$

2. Both $e_{dr}$ and $e_{qr}$ are expressed along each rotor axis of the equivalent circuit. For instance, along the rotor direct axis the impressed voltage is

$$\frac{1}{2}\left[\left(\frac{e_{dr} + je_{qr}}{f-v}\right) + \left(\frac{e_{dr} - je_{qr}}{f+v}\right)\right] = \frac{fe_{dr} + jve_{qr}}{f^2 - v^2} \tag{3.8}$$

The fact that a voltage $e_{dr}$ impressed along the rotor *direct* axis of the actual machine appears also as part of an impressed voltage along the quadrature axis of the equivalent circuit should not come as a surprise. This last voltage is $v/jf = v/p$ times as large as that impressed along the direct axis of the equivalent circuit. It seems that, whereas in the actual machine a rotor flux produces, say, an induced voltage ($jfB$) along the **d** axis and a generated voltage ($vB$) along the $\mathbf{q}_r$ axis, *in the cross-field equivalent circuit this role of the rotor fluxes is taken over by the rotor impressed voltages.*

The reason for retaining the phase shifter is that now *the currents in the equivalent circuits are identical with those flowing in the actual machine along the* **d** *and* **q** *axes.* The advantage of this is that any complex stationary network that is directly connected to the machine **d** and **q** axes is also connected unchanged to the equivalent circuit. The disadvantage is in the appearance of a phase shifter in the rotor.




(a) Synchronous machine

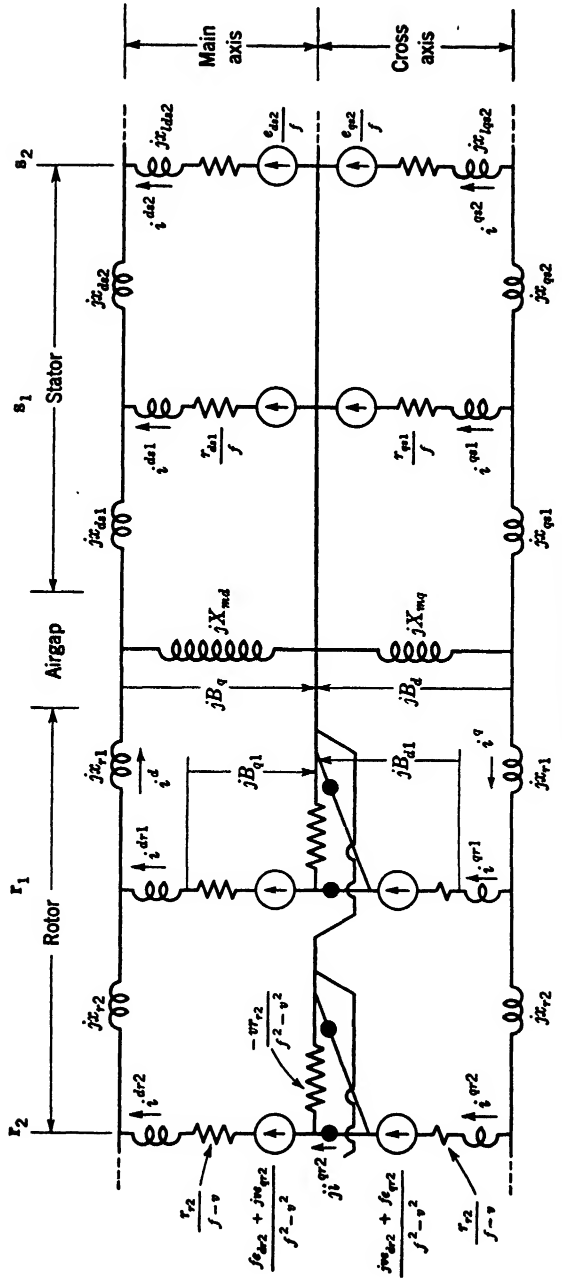
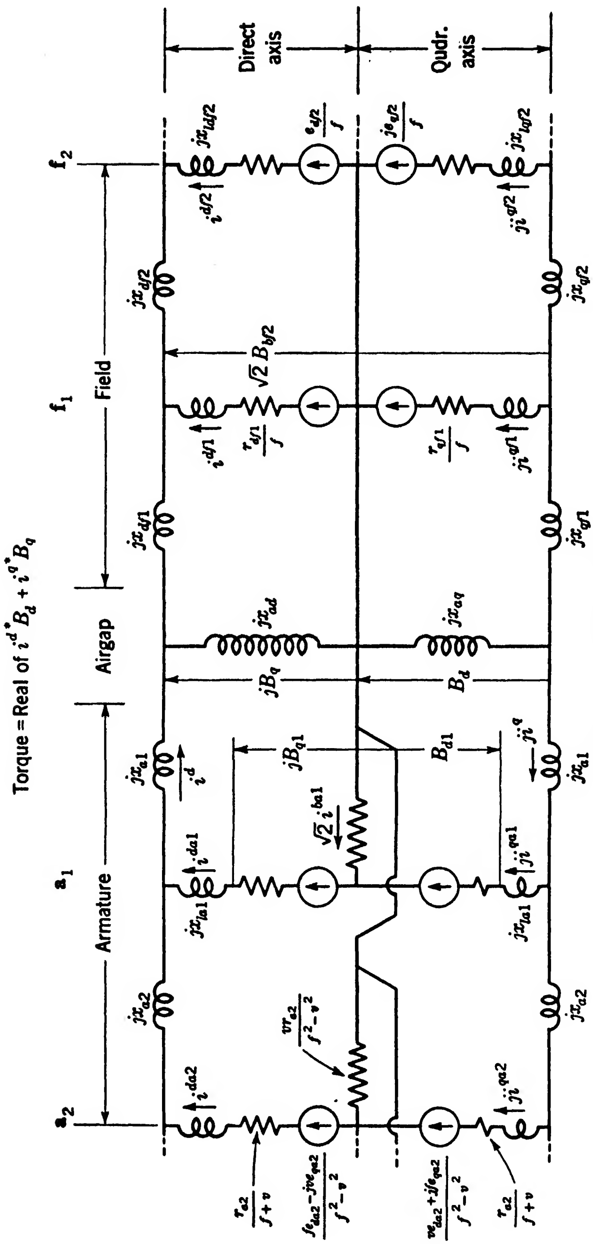
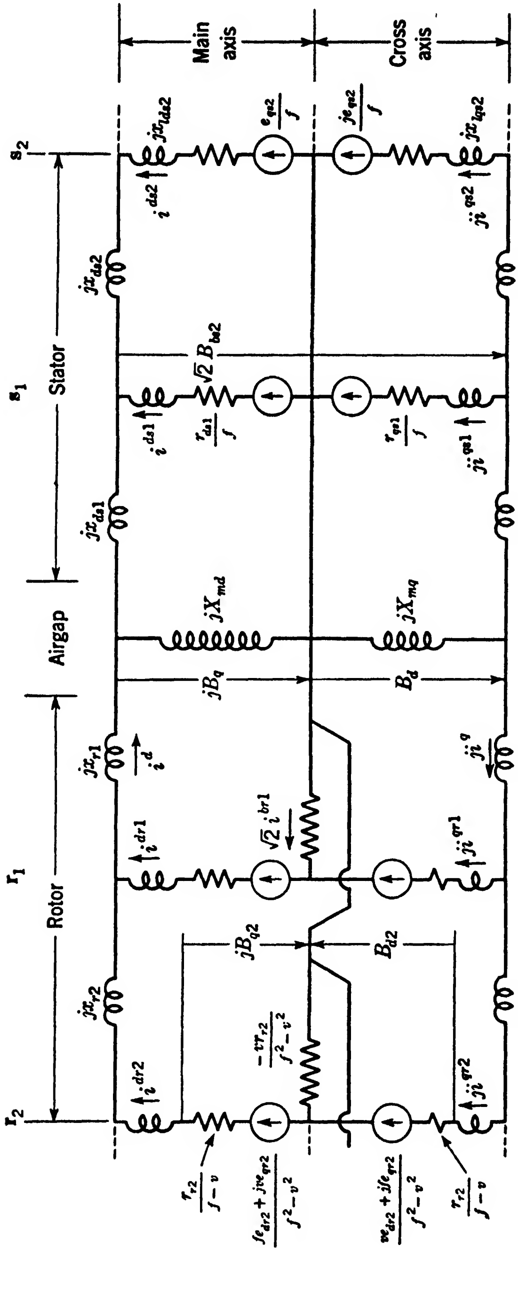



FIG. 3.5. The primitive machine rotating. (Cross-field theory. With phase shifters.)




(a) Synchronous machine




(b) Induction machine

FIG. 3.6. The primitive machine. (Cross-field theory. No phase shifters.)

When a rotating machine is connected to a simpler network, or to no network at all, the phase shifters in the rotor meshes of the primitive machine may be eliminated, as will be shown presently.

## CONSTANT TORQUE CALCULATIONS

The constant torque along the physical axes is still expressed as the scalar product $\mathbf{i}^*\mathbf{B}$.

$$T = \text{Real of } i^{dr*}B_{dr} + i^{qr*}B_{qr} \tag{3.9}$$

However, the voltages across the inductors no longer represent $\mathbf{B}$ but $j\mathbf{B}$ (or $j\boldsymbol{\phi}$). Expressing the above formula in terms of the measurable $jB$,

$$T = \text{Real of } -j[i^{dr*}(jB_{dr}) + i^{qr*}jB_{qr}] \tag{3.10}$$

## PRIMITIVE NETWORKS WITH NO PHASE SHIFTERS

Returning to Eq. 3.6 with the non-reciprocal mutuals, let the second possibility be followed. That is, let Eq. 3.6 be rewritten by assuming $ji^q$ and $je_q$ to be currents and impressed voltages, instead of $i^q$ and $e_q$.

$$\begin{aligned} e_d &= (Z_f + Z_M)i^d - Z_M(ji^q) \\ je_q &= -Z_M i^d + (Z_f + Z_M)(ji^q) \end{aligned} \tag{3.11}$$

The mutuals are now reciprocals.

*Hence if $ji^q$ is assumed to flow in all* **q** *meshes of the equivalent circuit instead of $i^q$, no phase shifters are necessary,* as shown in Fig. 3.6. The disadvantage of these circuits is that the currents in them are not the same as the currents in the machine, and a network connected, say, to the stator of an induction machine may not have a mutual impedance between the **d** and **q** meshes. If it has, the phase shifter reappears in the stator of the model.

It is also possible to consider Fig. 3.6 to be identical with Fig. 3.5, from which *the phase shifters have been moved out to the right.* All currents and voltages passed through have been multiplied by $j$. The voltages across the inductors represent now flux linkages ($jB_q = -j\phi_d$) in the **d** meshes and flux densities ($B_d = +\phi_q$) in the **q** meshes. Accordingly, the torque is

$$\begin{aligned} T &= \text{Real of } i^{dr*}B_d + i^{qr*}B_q \\ &= \text{Real of } i^{dr*}B_d - j(i^{qr*}jB_q) \end{aligned} \tag{3.12}$$

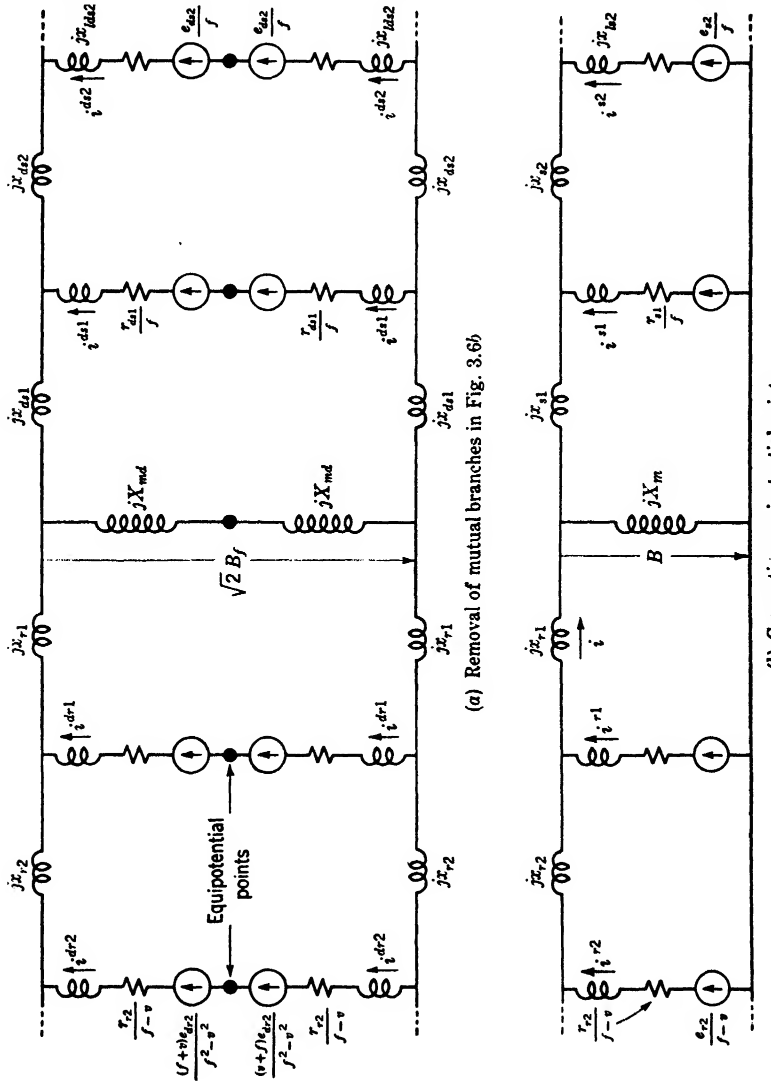
## SEQUENCE QUANTITIES IN THE PHYSICAL NETWORKS

It was shown in Chapter 2 that in the sequence equivalent circuit the common-branch current represents $\sqrt{2}ji^q$, and a voltage across both

meshes represents $\sqrt{2}jB_q$. A similar situation arises in the last cross-field equivalent circuit containing $ji^q$ (Fig. 3.6). Now, since $i^b = (i^d - ji^q)/\sqrt{2}$, this common-branch current is equal to $\sqrt{2}i^b$, and the voltage across both types of meshes is equal to $\sqrt{2}B_b$.

## THE PRIMITIVE POLYPHASE NETWORK OF THE CROSS-FIELD THEORY

In balanced polyphase machines (in which all impressed voltages are balanced and all structures smooth), whatever the currents and volt-



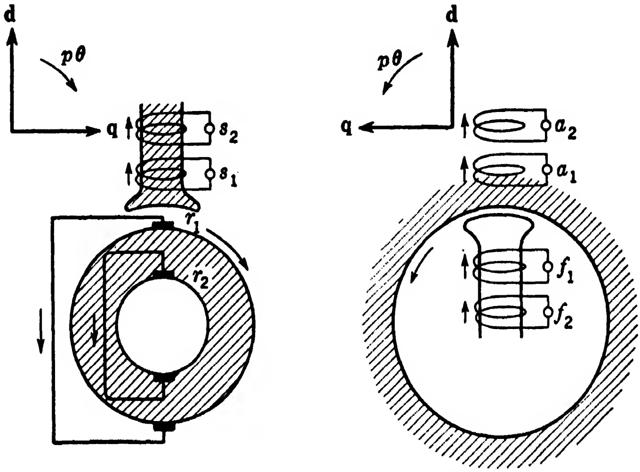

(a) Removal of mutual branches in Fig. 3.6b

(b) Connecting equipotential points

FIG. 3.7. Derivation of the primitive polyphase induction machine (cross-field theory).

ages that appear in the direct-axis windings, the same quantities appear in the quadrature-axis windings 90° later. That is, in polyphase machines, $i^q = -ji^d$ or $ji^q = i^d$. Similarly, in the equivalent circuit of Fig. 3.6 all such expressions as $je_q$ may be replaced by $e_d$, and $jB_q$ by $B_d$.

The reduced circuit is shown in Fig. 3.7*a*. It consists of two identical parts with no mutual branches, as the currents in the latter are zero.




(*a*) Induction (*b*) Synchronous

FIG. 3.8. The primitive polyphase machines.

Because of the symmetry, the midpoints of the vertical lines are equipotential points and may be connected together. *The resultant network, shown in Fig. 3.7b, is identical with that of the revolving-field theory* shown in Fig. 3.2, except that $i^d$ and $e_d$ replace $i^f$ and $e_f$, respectively.

Since $i^f = \sqrt{2}i^d$ and $e_f = \sqrt{2}e_d$, the voltages and currents in the cross-field primitive network are smaller, giving thereby only one-half of the total machine torque (that is, they give "torque per phase").

The primitive polyphase machine itself may thereby be represented by a set of four coils along the direct axis, as shown in Fig. 3.8.

# 4 THE TRANSFORMATION OF REFERENCE FRAMES

## CLASSIFICATION OF THE REFERENCE FRAMES OF THE PRIMITIVE MACHINE

The physical reference axes, hitherto assumed upon the stator and rotor of the primitive machine, all are rigidly connected either to the stator structure (primitive induction machine, Fig. 4.1II*b*) or to the rotor structure (primitive synchronous machine, Fig. 4.1II*a*), depending upon which structure has unbalanced windings. Later on (in Chapter 10) another type of physical reference frame will be introduced, in which the stator axes will be rigidly connected to the stator structure and the rotor axes will be rigidly connected to the rotor structure (Fig. 4.1I). This new type of reference frame arises when both stator and rotor windings are unbalanced.

In interconnecting two or more rotating machines, still other types of physical reference axes must be used and will be used later on (in Chapter 9) in connection with the primitive machine. These new axes will not be rigidly connected to either structure but will be free to rotate at speeds different from those of the rotor (Figs. 4.1III and IV). Of course, even on one machine any combination of fixed or free axes may appear.

Now with each combination of fixed and free *physical* axes assumed upon the actual primitive machine, it is possible to associate a *hypothetical* set of sequence axes, doubling thereby the number of possible reference frames of the primitive machine. It follows that in defining the concept of "sequence" axes it is absolutely necessary to define also the particular physical reference frame with which the sequence axes are associated.

It should be recalled that, along the sequence axes, in each mesh of the rotating primitive machine all currents and fluxes were of "absolute frequency." In setting up the primitive machine equivalent circuit, all quantities assumed "unit" frequency (namely, the frequency at which all reactances $X = 2\pi kL$ were calculated in each of the meshes). *This change in frequency may be looked upon as a transformation of reference frames* from the actual machine to its equivalent circuit. In general, as

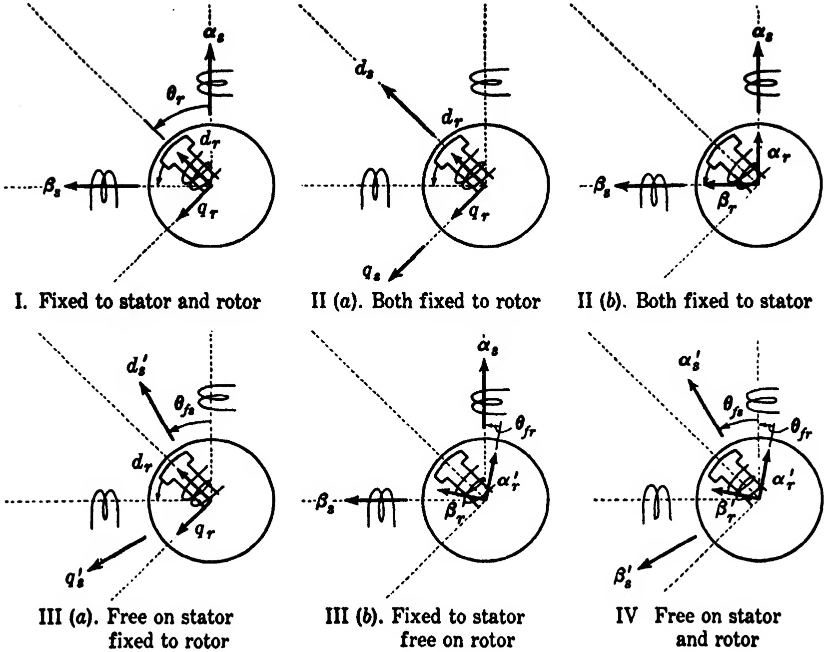



FIG. 4.1. Six physical, rectangular reference frames of the primitive synchronous machine.

TABLE 4.1 RECTANGULAR REFERENCE FRAMES OF THE PRIMITIVE MACHINE

| | Attachment of Stator and Rotor Reference Frames | Actual Machine | | Equivalent Circuit | |
|---|---|---|---|---|---|
| | | Physical | Hypothetical | Physical | Hypothetical |
| I | One fixed to rotor, one to stator | 1 | 7 | 13 | 19 |
| II | Both fixed to rotor | 2 | 8 | 14 | 20 |
| | Both fixed to stator | 3 | 9 | 15 | 21 |
| III | One fixed to rotor, one free | 4 | 10 | 16 | 22 |
| | One fixed to stator, one free | 5 | 11 | 17 | 23 |
| IV | Both free | 6 | 12 | 18 | 24 |

many types of equivalent circuits are possible as there are physical and sequence reference frames.

All these possible frames of the primitive machine are tabulated in Table 4.1.

The reference frames hitherto considered belong to group II. With each of these twenty-four basic reference frames, used most often in analytical studies, numerous auxiliary transformations may be introduced. For instance, in Fig. 3.6 all $i^q$ has been replaced by $ji^q$, representing new frames auxiliary to Nos. 14 and 15.

## REFERENCE FRAMES OF SYNCHRONOUS AND INDUCTION MACHINES

*The step from the primitive machine to any particular industrial machine is also a "transformation of reference frame."* It is the purpose of the present book to employ exactly the same step in going from one equivalent circuit to another as is employed in going from one actual machine to another.

It is important to point out that the two industrially most important classes of machines (the induction and synchronous machines) require the simplest possible types of transformations. *The step from the primitive to most actual induction or synchronous machines consists simply in opening one or more stator meshes in the equivalent circuit of the primitive machine.* The main difference between induction and synchronous machines lies merely in the location of the impressed voltages and in the values of the impressed frequencies $f$. It is this very simplicity of most transformations that led originally to the formulation of the primitive machine and decided the most practical form for this machine.

There are, however, some types of induction machines in which more complex transformations are needed. Most notable are the capacitor motor, in which a change in the number of turns of one stator winding is needed, and the shaded-pole motor, in which a stator winding is shifted by a constant angle, $\alpha$, in addition to a change in the ratio of turns.

## REFERENCE FRAMES OF COMMUTATOR MACHINES

Polyphase commutator machines involve other types of transformations besides a change in turn ratio and a shift in the angular position of a winding. Although the series connection of two stator or two rotor windings lying along the same axis is a self-evident type of transformation, the series connection of one *stator* and one *rotor* winding requires the introduction of new physical concepts, when equivalent circuits are to be established for them. These concepts will be given in Chapter 7.

In single-phase commutator machines some of the *rotor* windings are missing. Additional transformations arise when a *direct-axis* stator

winding is connected in series with a *quadrature-axis* rotor winding. The systematic treatment of the equivalent circuits of single-phase commutator machines used in control systems, such as the amplidyne, the rototrol, and, in general, the various metadynes invented by Pestarini, is deferred to another occasion.

## REFERENCE FRAMES OF INTERCONNECTED MACHINES

The interconnection of entire machines poses the question of *transformation from stationary to rotating reference frames,* since often the actually rotating leads of the rotors are connected to stationary leads of stators. Also it will often be found more judicious to introduce *reference frames rotating with the impressed fluxes* and not with the rotor. As a result, in interconnected systems the reference frame on each machine will have a different velocity and, in general, this velocity will be different from that of the rotor.

To keep track of the different velocities, an auxiliary phase shifter with a *variable* angle $\theta$ will be introduced temporarily for machines with slip rings in Chapter 9. These, however, will disappear from the final equivalent circuits.

## REFERENCE FRAMES OF STATIONARY NETWORKS CONNECTED TO A MACHINE

Stationary networks may be connected to a primitive machine (or actual machine) in two different manners:

1. The stationary networks are connected directly to the physical **d** and **q** axes. Such a situation occurs in the stator of induction machines (capacitor motor) and in the rotor of synchronous machines.

These cases are treated simply by adding impedances to the resistances and leakage reactances in the **f**, **b**, and common branches, as in the actual system.

2. The stationary networks are not connected directly to the assumed **d** and **q** axes. For instance, the networks may be connected to the slip rings of an induction motor or to the armature of a synchronous machine.

In these cases *the stationary networks must be first transformed to the axes of the rotating machine and then only connected to them.* The transformation of stationary networks to rotating axes will be treated in Chapter 8.

## TYPES OF EQUIVALENT CIRCUITS

If attention is shifted from the nature of the reference frames to the value of the impressed voltages, **e**, at least four types of networks may be defined:

1. *Steady-state* networks, in which both the magnitude and frequency of the impressed voltages are known.

2. *Self-excitation* networks, in which the magnitude of the impressed voltage is known to be zero, and the frequency of the current is unknown.

3. *Hunting* networks, in which the frequency of the impressed voltages is assumed to be known, and the magnitude of the voltages depends upon the steady-state network quantities.

4. *Constant flux-linkage* or *transient* networks, in which the instantaneous values of the flux linkages are assumed to be frozen and are employed as impressed voltages.

Examples will be shown of all these types of equivalent circuits, except for the hunting networks. The hunting networks will require a more detailed physical analysis.

## STEADY-STATE EQUIVALENT CIRCUITS

When the magnitude and frequency of the impressed voltages along the assumed reference frame are known, the equivalent circuits represent the steady-state performance of the machine. The value of the base frequency $f$ impressed along the assumed reference frame is also known and may be:

1. Zero in d-c and synchronous machines.
2. Unity in induction machines.
3. A variable quantity in commutator machines or in interconnected machines.

It is emphasized that the *frequency of the impressed voltages must be ascertained with respect to the physical* **d** *and* **q** *axes* and not in relation to axes along which the voltages are actually applied. For instance, if a voltage is impressed along the slip rings of an induction motor, its frequency must be examined at the hypothetical stationary brushes along which the reference frame of the equivalent circuit has been assumed.

It is also emphasized that in transforming from the physically existing **d** and **q** axes to the hypothetical sequence axes **f** and **b** *the base frequency* $p = jf$ *does not change.*

## SELF-EXCITATION EQUIVALENT CIRCUITS

Even though *no impressed voltages* exist, currents, nevertheless, may flow in a machine and also in its equivalent circuit. Such a condition is called self-excitation.

The unknown frequency, $f$, at which the machine self-excites may be found by assuming that, if *any* mesh is open-circuited and the impedance of the whole network is measured or calculated at the opened terminals,

the impedance of the whole network is zero. That is, in the equation $e = zi$ the current $i$ may assume a finite value, if $e = 0$ and $z = 0$. For an assumed rotor speed, $v$, the frequency of self-excitation may assume several values. Examples will be given at the ends of Chapters 6 and 7.

## HUNTING EQUIVALENT CIRCUITS

When a low-frequency ($h$) oscillation is superimposed upon the steady rotation of a rotor, the condition is called hunting. During hunting, one or two additional frequencies appear in the machine which require a separate equivalent circuit for each additional frequency. The fundamental-frequency and hunting-frequency circuits are electrically independent.

In the hunting networks the *magnitude* of the impressed voltages depends upon the steady-state values of the current and fluxes. In considering the impressed voltages, additional possibilities are introduced by the assumption that the reference axes themselves either take part in the oscillations or do not. Since the physics of the phenomenon becomes rather involved, the detailed study of hunting networks is not undertaken in the present volume. *However, it should be noted that all basic components of the hunting networks of any type of rotating machines have already been established here.* Only the *magnitude* of the impressed voltages (and currents) are left for further considerations.

## CONSTANT FLUX-LINKAGE EQUIVALENT CIRCUITS

The phenomenon of *sudden short circuit* may be represented approximately as a steady-state phenomenon at the *initial* stage of the short circuit and at the final *sustained* stage. The intermediate stage, namely, the decay of the initial quantities to their sustained state, may be treated, by the introduction of decrement factors, also as a succession of steady-state conditions.

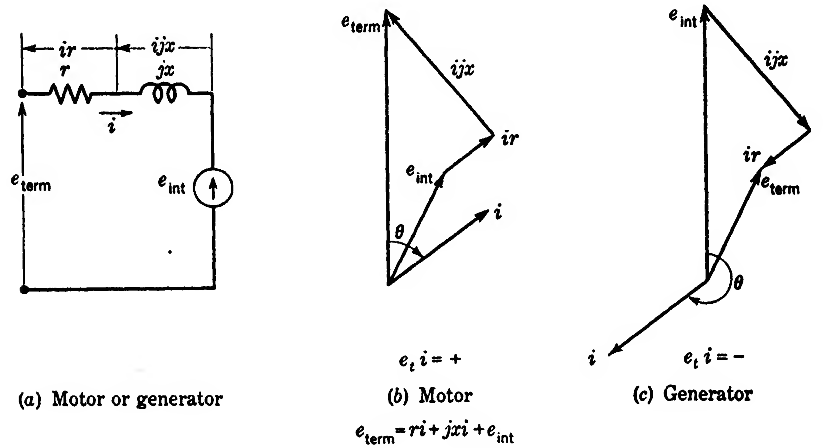
These equivalent circuits may or may not involve time harmonics. The impressed voltages consist of resultant flux linkages of the various windings. It is possible to calculate, for instance, the instantaneous and sustained fundamental and harmonic currents, as well as torques that exist when a salient-pole alternator is subjected to a single-phase or a three-phase short circuit. The equivalent circuits also allow the determination of all decrement factors by single measurements. Such circuits will be treated in detail in Chapter 12.

## SIGN CONVENTIONS IN THE EQUIVALENT CIRCUITS

The *sign conventions in this book have been defined to correspond to those of the dynamical equations of Lagrange.* In consequence, all equations

and equivalent circuits refer to a *motor*, that is, to a machine in which the entering current is called positive (Fig. 4.2*a*). The torque $T = iB$ (representing the electromagnetic torque on the *non-salient member*) comes out positive for a motor. Hence, in a generator the entering current assumes a negative value and the torque is a negative number.

In the equivalent circuit a voltage arrow shows a *rise* of potential. Hence $ir$ and $ijx$ point in the opposite direction of $i$. Also the arrow for




(*a*) Motor or generator (*b*) Motor (*c*) Generator

FIG. 4.2. Equivalent circuit sign convention.

the impressed voltage points in the same direction as the sum of the internal $ir$ and $ijx$ drop according to the defining equation

$$e_{\text{imp}} = ir + ijx + e_{\text{int}} \qquad 4.1$$

where $e_{\text{int}}$ is any internal generated voltage.

In drawing vector diagrams for the equivalent circuit, the arrow convention corresponding to this last equation is shown in Fig. 4.2*b* and *c* for a motor and a generator, respectively.

## SEVERAL FREQUENCIES OF CURRENTS

If the impressed voltages or the currents in the rotor have different frequencies from those impressed on the stator (when viewed from the stationary direct and quadrature axes), then *as many separate equivalent circuits must be established as there are frequencies of voltages (or currents). The equivalent circuits are electrically isolated from each other.*

For instance, if a 60-cycle voltage is impressed on the stator of an induction motor and also across its slip rings, then the frequency of the latter voltage measured from the stationary axes is not 60 but $(1 + v)$

60 cycles. Hence two separated equivalent circuits must be established: in one of them $f = 1$ (if the design constants $x$ have been calculated for 60 cycles); in the other $f = (1 + v)$, where $v$ is a variable quantity.

When two or more machines are interconnected, often currents with several frequencies appear (especially if the machines are unbalanced), even though the frequency of the impressed voltages is the same when viewed from stationary reference axes.

Whereas in the actual machine all the currents of different frequency flow in the same winding, in the equivalent circuit they flow in different networks, even though they now have identical (unit) frequency. The resultant torques assume all the possible sum and difference frequencies. The calculation of these torques is the next subject matter.

## PRODUCTS OF TWO COMPLEX NUMBERS

The currents and voltages of various frequencies are represented on the equivalent circuits as ordinary complex numbers that give, however, no clue to their frequency. For calculating torques of various frequencies by products of complex numbers ($T = iB$) certain routine rules must be established.

Let two complex numbers be given, one representing the rms value of a sinusoidal quantity with $\omega_1$ frequency in time, the other that of an $\omega_2$ frequency. (Their space variation is ignored.)

$$A + jB = \sqrt{2}(A \cos \omega_1 t - B \sin \omega_1 t) = \text{Real of } \sqrt{2}(A + jB)\epsilon^{j\omega_1 t}$$

$$C + jD = \sqrt{2}(C \cos \omega_2 t - D \sin \omega_2 t) = \text{Real of } \sqrt{2}(C + jD)\epsilon^{j\omega_2 t}$$

Their product consists of two sinusoidal waves in time. In particular:

1. A sum-frequency wave.

$$T_{+} = E \cos (\omega_1 + \omega_2)t - F \sin (\omega_1 + \omega_2)t$$

$$= \text{Real of } \frac{E + jF}{\sqrt{2}} \epsilon^{j(\omega_1 + \omega_2)t} \qquad 4.2$$

It is found simply by multiplying together the two complex numbers.

$$E + jF = (A + jB)(C + jD) = (AC - BD) + j(BC + AD)$$

2. A difference-frequency wave.

$$T_{-} = G \cos (\omega_1 - \omega_2)t - H \sin (\omega_1 - \omega_2)t$$

$$= \text{Real of } \frac{G + jH}{\sqrt{2}} \epsilon^{j(\omega_1 - \omega_2)t} \qquad 4.3$$

It is found by multiplying together the two complex numbers and then *taking the conjugate of the one whose frequency is subtracted.*

$$G + jH = (A + jB)(C + jD)^* = (AC + BD) + j(BC - AD)$$

The resultant complex number gives the *peak* value of the sine wave.

When both complex numbers have the same frequencies, their difference frequency is the constant or zero-frequency torque. This is the reason for the existence of conjugates in the constant-torque formulas. The imaginary part, $-F \sin (\omega_1 - \omega_1)t$, is zero; hence only the *real* part of the product is taken.

## ALTERNATING TORQUES ALONG PHYSICAL AXES

Let the rotor currents be $i^d$, $i^q$ of $\omega_1$ frequency and the rotor fluxes be $B_d$, $B_q$ of $\omega_2$ frequencies. Again the fluxes may be either the *resultant* rotor fluxes or the *airgap* fluxes, as the rotor leakage fluxes cannot produce torques with the rotor currents, if the rotor is smooth. Since $i^d$ cannot produce a torque with $B_q$, or $i^q$ with $B_d$, the two types of torques produced are as follows:

1. Sum-frequency torques.

$$T_+ = i^d B_d + i^q B_q = T' + jT'' \qquad 4.4$$

2. Difference-frequency torques.

$$T_- = i^{d*} B_d + i^{q*} B_q = T' + jT''$$

or $\qquad 4.5$

$$T_- = i^d B^*_d + i^q B^*_q = T' + jT''$$

depending on which frequency is being subtracted.

The important rule to be remembered is that, *if any of the torque frequencies, say $i^q B^*_q$, comes out negative, then the conjugate of that term must be taken* in order that the resultant product frequency should come out positive.

The peak value of several sinusoidal torques of the same frequency is found by

$$T_{\text{peak}} = \sqrt{\Sigma(T')^2 + \Sigma(T'')^2} \qquad 4.6$$

## ALTERNATING TORQUES ALONG SEQUENCE AXES

Replacing $i^d$ by $(i^f + i^b)/\sqrt{2}$, etc.,

1. The sum-frequency torques are

$$T_+ = i^f B_b + i^b B_f \qquad 4.7$$

2. The difference-frequency torques are

$$T_- = (i^f)^* B_f + (i^b)^* B_b$$

or  4.8

$$T_- = (i^f) B^*_f + i^b B^*_b$$

The great advantage of the sequence equivalent circuits is that the denominator of the resistances gives outright the "absolute" frequencies of each current and flux under consideration. In the physical equivalent circuits (along **d** and **q**), that is not the case, and the frequencies of the various currents and fluxes must be found along the **d** and **q** axes by physical considerations.

## TORQUES OF SEVERAL HARMONIC CURRENTS

When several harmonic currents flow in *one* winding, then all possible products of currents and fluxes must be considered. For instance, let

$$\begin{aligned} i^f &= i^{f1} + i^{f2} + i^{f3} + \cdots \\ B_f &= B_{f1} + B_{f2} + B_{f3} + \cdots \end{aligned} \qquad 4.9$$

with similar expressions for $i^b$ and $B_b$. Then the *sum-frequency* torque components are

$$\begin{aligned} T_+ &= i^f B_b + i^b B_f \\ &= (i^{f1} + i^{f2} + i^{f3} + \cdots)(B_{b1} + B_{b2} + B_{b3} + \cdots) \\ &\quad + (i^{b1} + i^{b2} + i^{b3} + \cdots)(B_{f1} + B_{f2} + B_{f3} + \cdots) \\ &= i^{f1} B_{b1} + i^{f1} B_{b2} + i^{f1} B_{b2} + \cdots \end{aligned} \qquad 4.10$$

Similar procedure applies for the calculation of the *difference-frequency* torques.

The absolute frequencies of $i$ and $B$ are simply read off the sequence equivalent circuits and added. The result is the *absolute frequency* of torque due to that term. In calculating the *magnitude* of torque due to that term, the two complex numbers of $i$ and $B$ are simply multiplied together, giving a third complex number. If the absolute frequency of a torque term is negative, then the conjugate of the product has to be taken.

## STEADY-STATE AND TRANSIENT VOLTAGE EQUATIONS

In Fig. 2.4*e* the *transient* equivalent circuit of the primitive machine is given for the case in which the machine is at rest. From this equivalent circuit the transient analytical equations can easily be written.

(There are as many equations as there are meshes.) The steps from this transient network of the primitive machine along the physical **d**, **q** axes (Fig. 2.4*e*) to the sinusoidal form of an *actual* machine along the hypothetical **f**, **b** axes (Fig. 3.1 and all the other figures to come) are as follows:

1. All $p = d/dt$ was replaced by $jf\omega$.
2. $\omega L$ was replaced by $x$ (Fig. 2.5*a*).
3. Symmetrical components were introduced by means of a set of scalars (Fig. 2.5*b* and *c*).
4. The resistance and impressed voltages were divided by the absolute frequencies (a set of scalars) containing $f$ and the rotor speed $v = p\theta$ (Fig. 3.1).
5. The primitive machine will be changed by a transformation to some actual machine (all figures to follow).

If, then, the equivalent circuit of the primitive machine or any other machine has been developed for some sinusoidal operating condition by the method of the present book, then *the sinusoidal and transient equations of performance of the machine immediately may be established in their conventional form* by first writing down the equation of the final revolving-field equivalent circuit and then retracing some of the above steps. In particular:

1. The *sinusoidal* performance of the machine along the *sequence* axes may be found by multiplying each equation by its respective absolute frequency $n$ (the denominator of the resistance term).
2. The *transient* performance of the machine along the sequence axes is found by the additional steps of replacing $f$ by $-jp = p/j = (d/dt)/j$ and replacing $x$ by $L$.
3. The *transient* performance of the machine along the *physical* axes may be found by transforming back the above transient sequence equations to the physical axes by Eq. 3.5*a*.
4. The *sinusoidal* performance of the machine along the *physical* axes is found either by transforming step 1 by Eq. 3.5*a* or, if step 3 is available, by replacing $p$ by $jf$ and replacing $L$ by $x$.

When several machines are interconnected and time harmonics exist, nevertheless *only one base frequency $f$ occurs in all the meshes of the equivalent circuit;* hence the replacement of $f$ by $p/j$ is unmistakable.

In writing down the equations of voltage it has to be remembered that each mesh passes through the airgap reactance and through *one* of the vertical resistances whose denominator gives the absolute frequency. It also should be noted that in establishing the analytical equations the cross-field equivalent circuits were not mentioned since their use requires an additional step. The use of the cross-field equivalent circuits

would nevertheless lead to *correct* equations, though the equations would not assume the conventional form.

In acceleration problems all $v$ are replaced by $p\theta = d\theta/dt$.

## STEADY-STATE AND TRANSIENT TORQUE EQUATIONS

If the differences of potential $B$ in the circuits, representing the flux densities in the torque expression, Eqs. 4.7 and 4.8, are expressed in terms of the currents and reactances of the equivalent circuit (and if $x$ is replaced by $L$), an analytical expression for the electrical torque is found outright for both sinusoidal and transient constant-speed performance.

For acceleration studies the mechanical torque of the rotor $T_M = p^2 I$ is added to the torque expression, where $I$ is the moment of inertia of the rotor.

An example is given in Appendix 1 for the determination of the *transient* equations of voltage and torque of a salient-pole synchronous machine running at an arbitrary speed.

## UNBALANCED QUANTITIES

When the word "unbalanced" occurs in the text, careful attention has to be paid to the quantity that is supposed to be unbalanced. The following unbalanced quantities will occur:

1. Unbalanced *windings*. The resistance $r_q$ and leakage reactance $jx_q$ of a winding along the quadrature axis may differ from $r_d$ and $jx_d$ of a winding along the direct axis, even though both windings belong to the same layer. (For instance, the two windings in the stator of a split-phase induction motor are unbalanced.)

2. Unbalanced *airgap*. The stator or rotor has salient poles, so that $X_{md}$ differs from $X_{mq}$, where $X_m = X_g$ is the mutual reactance between a stator and rotor winding.

3. Unbalanced *load*. An outside impedance connected to the quadrature axis winding differs from that connected to the direct-axis winding (of the same layer of winding).

4. Unbalanced *impressed voltage*. Both forward and backward components $e_f$ and $e_b$ are impressed on one layer of winding.

# 5 INDUCTION MACHINES

## STEPS FROM THE PRIMITIVE TO ACTUAL MACHINES

Although the base frequency $k$ at which the design constants $x = 2\pi kL$ have been calculated (or tested) is usually the same as the frequency $f$ of the impressed stator voltage—both being unity—in the equivalent circuits it will be assumed that *the frequency $f$ of the impressed voltage is variable.* In most practical cases, when the design constants have been calculated, say, at 60 cycles, and the impressed frequency is also 60 cycles, $f$ in the equivalent circuits becomes unity.

The ways in which industrial machines characteristically differ from the hypothetical primitive machine will be considered as various types of "transformations." Each type of transformation is separately treated in the order of increasing complexity, so that *complicated transformations will be built up of a series of simpler transformations.* For all induction motors, both the revolving-field (sequence-axis) and cross-field (direct- and quadrature-axis) equivalent circuits will be given.

## REMOVAL OF LAYERS OF WINDING

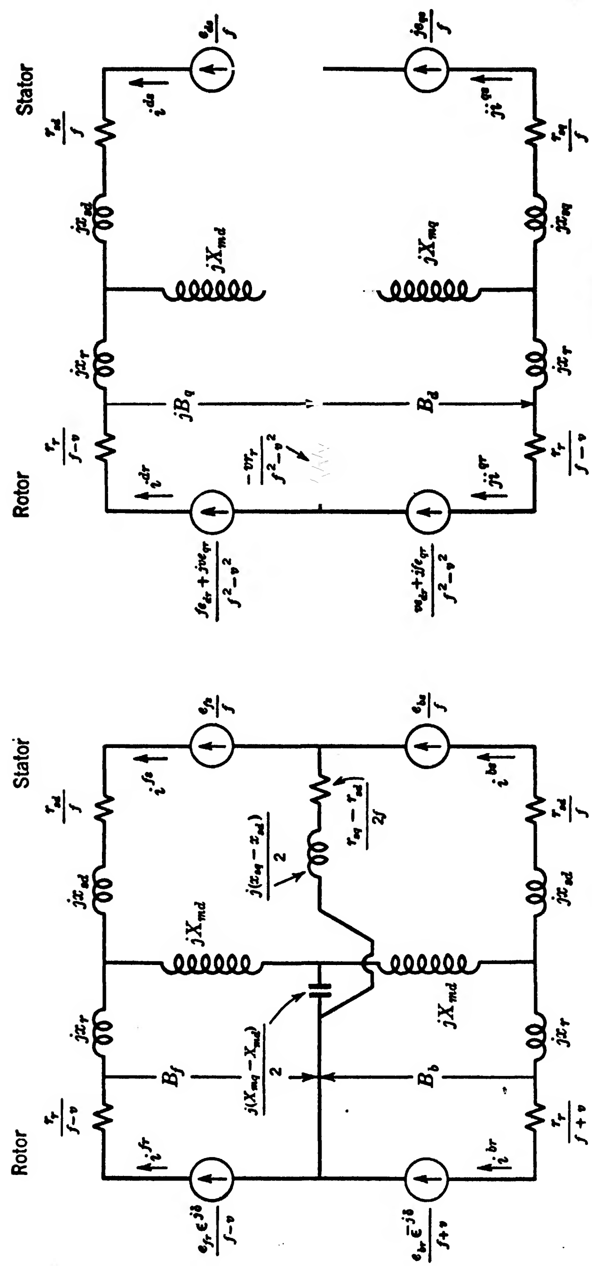
The simplest possible "transformation" through which a practical industrial machine differs from the hypothetical primitive machine is the absence of entire layers of windings from the stator and rotor. The absence of one layer of winding is represented on both the cross-field and revolving-field equivalent circuits simply by removing two neighboring **d** and **q** (or **f** and **b**) meshes from the primitive network of Fig. 3.1*a*.

When more than two layers of windings exist on one of the structures of the actual machine, the manner of *addition* of sets of meshes to the primitive equivalent circuit is self-evident.

## UNBALANCED DOUBLY FED INDUCTION MOTOR (FIG. 5.1) *

One of the most complex forms of induction motors has a salient pole (either on the stator or on the rotor), has one layer of unbalanced wind-

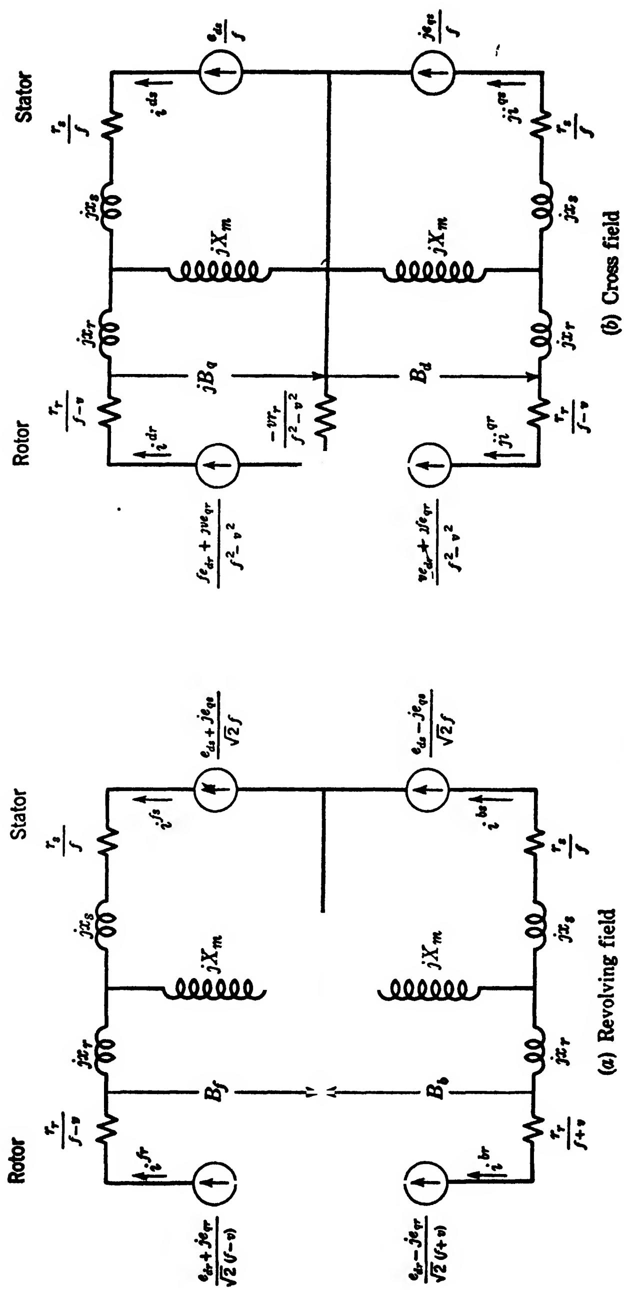
* Gabriel Kron, "Steady State Equivalent Circuits of Synchronous and Induction Machines," *Transactions of the AIEE*, Vol. 67, pp. 175–181, 1948.




(a) Revolving field

(b) Cross field

FIG. 5.1. Doubly fed unbalanced induction motor.




(a) Revolving field

(b) Cross field

FIG. 5.2. Doubly fed standard induction motor.

ings on its salient structure, and may have unbalanced voltages (both **f** and **b**) impressed on all stator and rotor windings. A variable-frequency excitation is also assumed for greater generality.

## DOUBLY FED STANDARD INDUCTION MOTOR WITH UNBALANCED VOLTAGES (FIG. 5.2)

If a *standard* three-phase induction motor is excited, all mutual couplings between the meshes **f** and **b** disappear. Both stator and rotor may have unbalanced voltages with variable frequencies. Along the **d** and **q** axes the frequencies must be identical on both stator and rotor. Hence along the slip rings the frequency of the forward rotor impressed-voltage if $f - v$ (slip frequency) and that of the backward rotor impressed-voltage is $f + v$.

## POLYPHASE MACHINES

A polyphase machine is defined as one with balanced windings, with smooth airgap, and with only a positive-sequence voltage impressed. The absence of layers of winding is represented on the primitive polyphase equivalent circuit by removing *one* mesh for each absent layer of winding. The cross-field and revolving-field equivalent circuits of polyphase machines (Figs. 3.7 and 3.2) differ in two respects from each other:

1. One has a subscript $f$; the other has $d$.
2. The revolving-field network represents the whole machine; the other, one phase of it only.

Since the difference is insignificant, only the revolving-field (sequence) networks will be discussed.

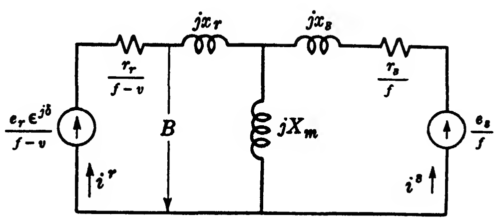



FIG. 5.3. Doubly fed polyphase induction motor.

## DOUBLY FED POLYPHASE INDUCTION MOTOR (FIG. 5.3)

If only positive-sequence voltages are impressed on both stator and rotor with a given angular displacement $\delta$ between the two voltages, then only the upper part (forward meshes) of the revolving-field network of Fig. 5.2 remains.

## STANDARD POLYPHASE INDUCTION MOTOR (FIG. 5.4)

If in the previous network the rotor excitation is absent, the well-known standard equivalent circuit is left. (When the reference axes rotate at various speeds, the resultant equivalent circuits appear in

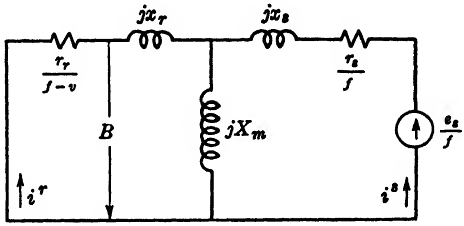



FIG. 5.4. Standard polyphase induction motor.

Chapter 9.) Note the appearance of the variable-frequency symbol $f$. This important factor is absent in conventional circuits.

## DOUBLE SQUIRREL-CAGE INDUCTION MOTOR (FIG. 5.5)

If in the primitive polyphase machine both rotor meshes and only one of the stator meshes are retained, Fig. 5.5 results.

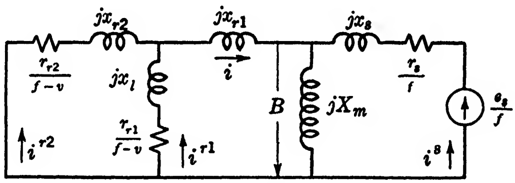



FIG. 5.5. Double squirrel-cage induction motor.

## MULTIPLE SQUIRREL-CAGE INDUCTION MOTOR (FIG. 5.6)

It is possible to add several additional rotor meshes, all analogous to the second mesh. The parallel rotor branches in the equivalent circuit may be replaced by a single impedance.

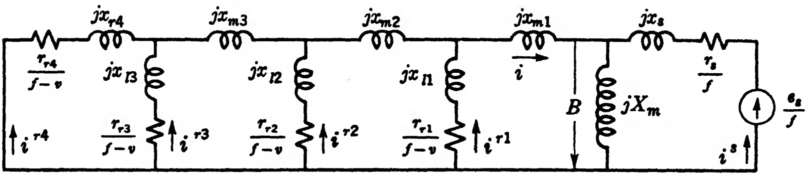



FIG. 5.6. Multiple squirrel-cage induction motor.

## UNBALANCED THREE-PHASE NETWORKS

Three unequal stationary impedances with unequal but reciprocal mutual impedances between them may always be transformed from the three-phase axes **a**, **b**, **c** to the two-phase axes **d**, **q**, **o**. The zero-sequence impedance is reciprocally coupled to the **d** and **q** axes. The **d** and **q** axes themselves are also coupled reciprocally.

In this general case it is preferable to leave the stationary networks and their connected rotating machine equivalent circuits along the **d** and **q** axes, since a change to the **f**, **b** axes introduces non-reciprocal mutuals.

When the unbalanced three-phase impedances are symmetrical with respect to one of the phases, then both the **d**, **q**, **o** and the **f**, **b**, **o** axes are reciprocally coupled.

Since the representations of the unbalanced three-phase stationary networks along the **d**, **q**, **o** (or rather $\alpha$, $\beta$, **o**) and along the **f**, **b**, **o** axes are well known, they are not considered in this book. As mentioned before, the zero-sequence axes **o** and their couplings to the **d**, **q** or **f**, **b** axes will be ignored, as they can always be added *afterward* to the unbalanced two-phase equivalent circuits to be developed for the rotating machines proper.

## STATIONARY NETWORKS ALONG STATIONARY REFERENCE FRAMES

When a stationary network is connected to the slip rings of a wound *rotor*, the network appears to be rotating if viewed from the stationary

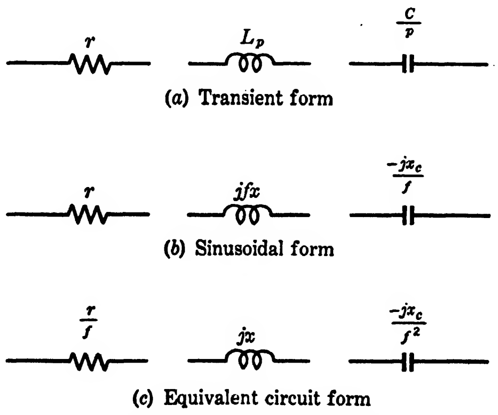



FIG. 5.7. Stationary impedances along stationary axes.

**d** and **q** axes of the rotor. The theory of such stationary networks is considered in Chapter 8. If, however, the stationary network is connected to the *stator* winding of an induction motor, its theory becomes

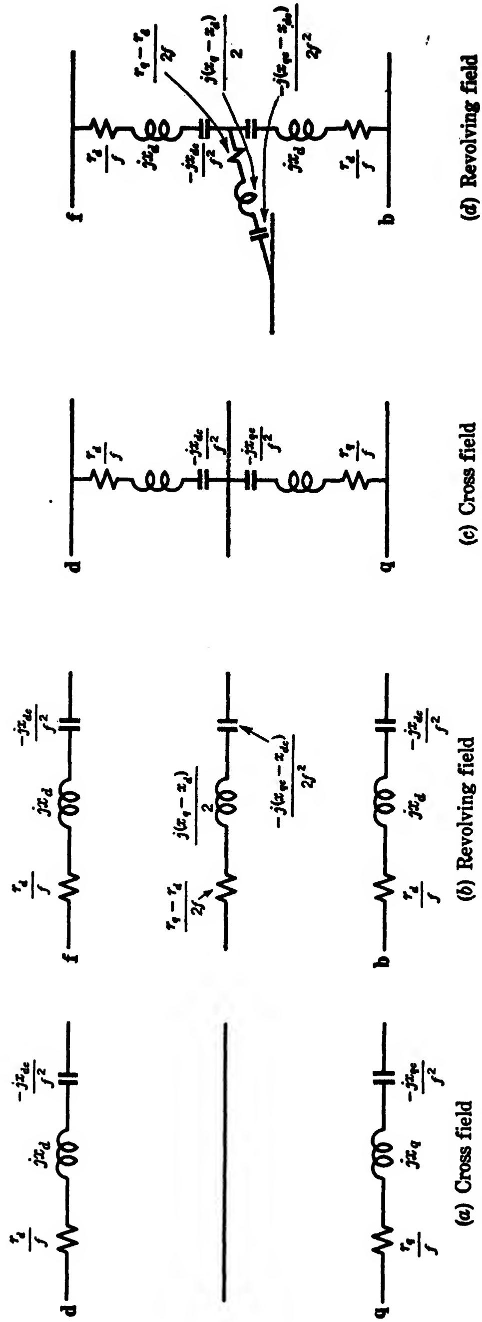



FIG. 5.8. Unbalanced two-phase networks along stationary axes.

the same as that of the stator resistance and leakage inductance. Since the connected network may include capacitors also, the theory is reviewed here again.

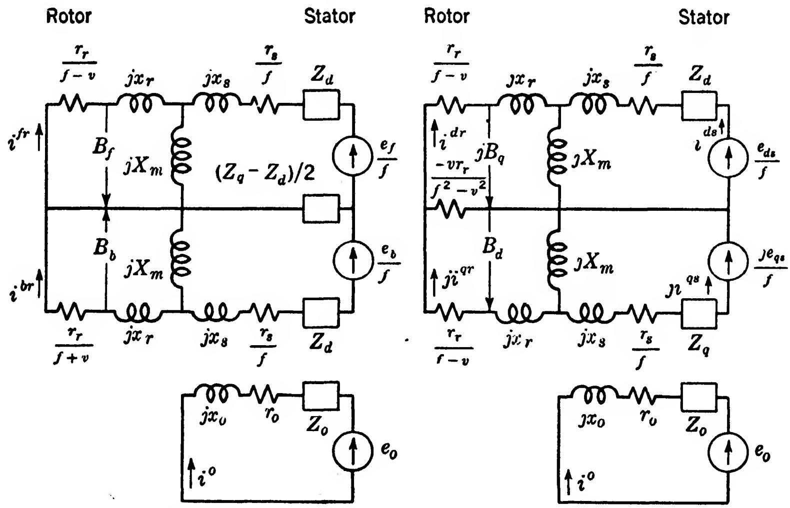
The transient form of an isolated resistor, inductor, and capacitor is indicated in Fig. 5.7*a*. When the frequency of the current flowing through them is $f$ (that is, if $p = jf\omega$), they assume the form shown in Fig. 5.7*b*. Since in the equivalent circuit all stator impedances are divided by the absolute frequency $f$, the equivalent circuit form of the three impedances is given in Fig. 5.7*c*. It should be noted that *the capacitor is divided by* $f^2$.

Let any combination of series and shunt networks exist in each axis with no mutual coupling between the **d** and **q** circuits (Fig. 5.8*a* and *c*). (The effect of mutual coupling is considered later on in this chapter.) When the stationary network along the quadrature axis differs from that along the direct axis, the cross-field equivalent circuit is given in Fig. 5.8*a* and *c*. Their revolving-field form, Fig. 5.8*b*, follows that of any other unbalanced stator winding, and Fig. 5.8*d* follows one in which the airgap reactance is unbalanced.

These networks may now be interconnected with the *stator* windings of induction motors, but not with their wound rotors.

## INDUCTION MOTORS WITH UNBALANCED STATOR LOADS (FIG. 5.9)

When standard balanced three-phase induction motors are connected to outside impedances that may be unbalanced, the impedances can




(*a*) Revolving field (*b*) Cross field

FIG. 5.9. Induction motors with unbalanced stator inpedances.

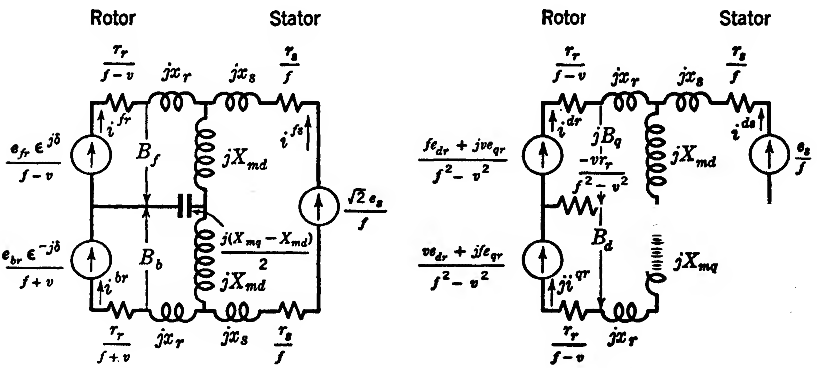
always be replaced by some combination of series and shunt loads plus a zero-sequence network. When the three-phase windings themselves are also interconnected, the resultant equivalent circuits are considered at the end of this chapter.

## REMOVAL OF THE QUADRATURE-AXIS WINDING

The above motors were developed from the primitive machine by removing layers of windings from this machine. Such a removal was self-evident in both cross-field and revolving-field networks.

A single-phase machine may be assumed to be derived from a *two-phase* machine by removing (or open-circuiting) the quadrature-axis winding of the stator in the two-phase machine. (A *three-phase* machine is usually changed to a single-phase machine by leaving one phase open-circuited and connecting the other two phases in series. The axis of the series winding is called now the "direct" axis.) In the cross-field network the stator q mesh is removed in an analogous manner, but it is not immediately obvious how to take care of such a "transformation" in the revolving-field network.

It should be recalled from Fig. 3.1 that the common branch has $\sqrt{2}ji^{qs}$ flowing in it. Hence, *when $i^{qs}$ is 0, the corresponding common branch in the revolving-field network is opened.* The current flowing in the single stator mesh is $i^{fs}$ or $i^{ds}/\sqrt{2}$. Similarly, the impressed stator voltage is $e_{fs} + e_{bs}$ or $\sqrt{2}e_s$, where $e_s$ is the voltage impressed on the machine.




(a) Revolving field (b) Cross field

FIG. 5.10. Single-phase selsyn.

## SINGLE-PHASE SELSYN (FIG. 5.10)

The stator has a salient pole, and no winding exists on the quadrature axis. The rotor slip rings are connected to other systems. In practice,

usually the roles of the stator and rotor are interchanged and the salient pole rotates.

## STANDARD SINGLE-PHASE INDUCTION MOTOR (FIG. 5.11)

In a standard single-phase induction motor, the airgap is smooth and $x_{md} = x_{mq}$. Also this rotor has a squirrel-cage winding on it; hence it is permanently short-circuited along all axes.

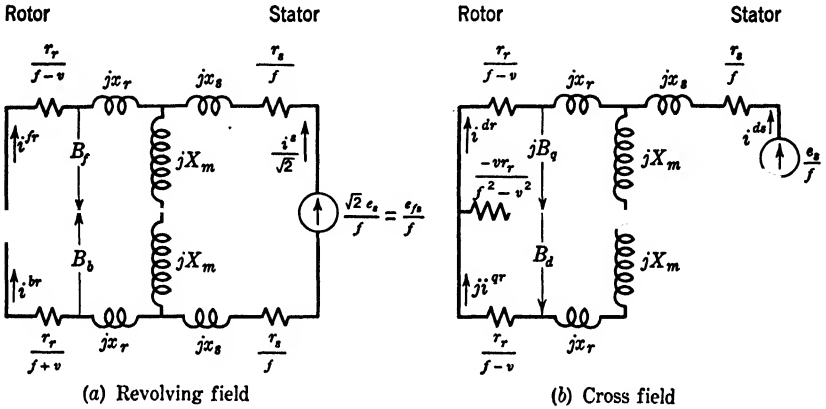



FIG. 5.11. Single-phase induction motor.

## CHANGING THE RATIO OF TURNS

In setting up the equivalent circuit of the primitive machine one of the assumptions was that all windings of the primitive machine have the same number of turns. When the ratio of turns becomes different from unity, the primitive network must be generalized, as will be shown in Chapter 7.

When only one out of two stator windings of one layer has a different number of turns, the equivalent circuit may be left unchanged. Instead it may be assumed that *the current and voltage existing in the particular mesh of the network differs by the ratio of turns from those existing in that particular winding of the machine.*

Let

$$a = \frac{\text{Number of turns of quadrature-axis winding}}{\text{Number of turns of direct-axis winding}}$$

Then it is assumed that the network $i$ represents not the current in the machine but the mmf. Hence:

1. If the current flowing in the machine is $i^q$, then $ai^q$ flows in the network.

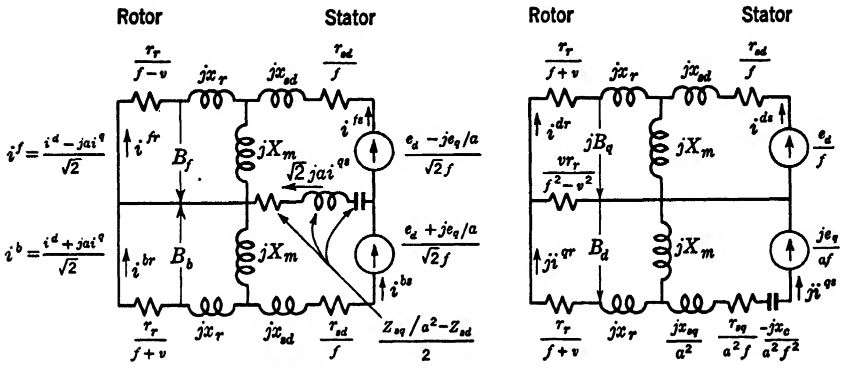
2. If the voltage impressed on the machine is $e_q$, then $e_q/a$ is impressed on the network.

3. If the actual leakage impedance of the winding is $z$, then $z/a^2$ appears in the network (since this primitive network is still wound with unit ratio of turns).

## CAPACITOR (AND SPLIT-PHASE) MOTOR (FIG. 5.12) *

The capacitor (and the resistance split-phase) induction motor differs in three respects from the primitive machine:

1. Its direction of rotation, $v$, is opposite (counterclockwise).




(a) Revolving field (b) Cross field

FIG. 5.12. Capacitor motor.

2. It has an additional capacitor or resistance, $z_q$, connected in series with the stator quadrature-axis (or starting) winding.

3. The ratio of turns of the quadrature winding to the direct-axis winding is not unity but $a$.

These differences can be taken care of as follows:

1. In the equivalent circuit both the direction of rotation of the rotor, ($v$), and of the forward wave, (f), changes. (That is, $i^f$ is defined as $(i^d - ji^q)/\sqrt{2}$ instead of $(i^d + ji^q)/\sqrt{2}$.) The combined result is to change only the sign of $e_q$ in the equivalent circuit.

2. The effect of adding a load, $z_q$, in the quadrature axis is to add a mutual impedance term $z_q/2$ in the common stator branch of the revolving-field circuit. In the cross-field circuit $z_q$ is added only to the **q** branch.

3. Let the quadrature-axis quantities $r_{qs}$, $x_{qs}$, and $z_q$ be calculated with the winding having $a$ number of turns, and let the actual machine

* Gabriel Kron, "Equivalent Circuit of the Capacitor Motor," *General Electric Review*, Vol. 44, p. 511, September 1941.

voltage and current be $e_{qs}$ and $i^{qs}$. The equivalent circuit, however, retains all design constants, voltages, and currents, as if the quadrature axis had remained with unit number of turns.

Hence:

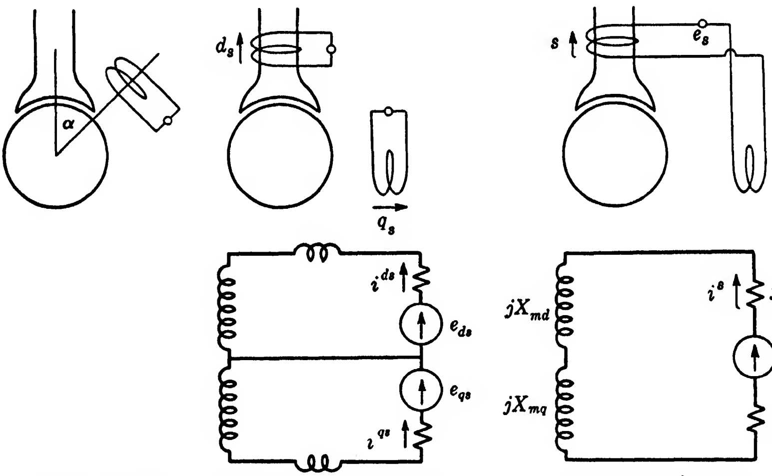
(*a*) In the machine $e^q$ is impressed; in the circuit $e_q/a$ is impressed.

(*b*) In the machine $i^q$ flows; in the circuit $ai^q$ flows (hence $i^{fs}$ is defined as $i^{fs} = [i^{ds} - jai^{qs}]/\sqrt{2}$).

(*c*) In the machine the design constants are $r_{sq}$, $jx_{sq}$, and $z_q$; in the circuit they appear as $r_{sq}/a^2$, $jx_{sq}/a^2$, and $z_q/a^2$.

## ONE STATOR WINDING AT AN ANGLE

When a stator winding is located at an angle $\alpha$ from the direct axis (Fig. 5.13*a*) it may be looked upon as a separate layer of winding consisting of one direct-axis winding and one quadrature-axis winding (Fig. 5.13*b*). If the number of turns on the winding is $a$, then the direct-



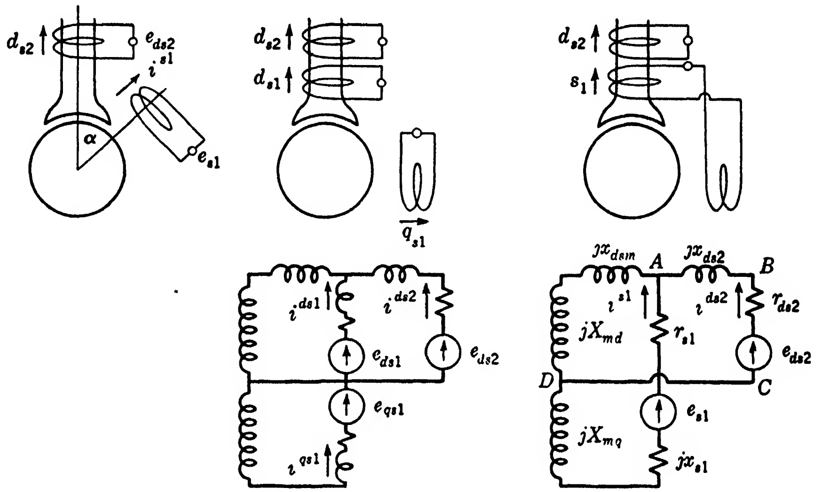

(*a*) Actual winding (*b*) Two equivalent meshes (*c*) Reactance definitions

FIG. 5.13. One stator winding at an angle.

axis winding has $a \cos \alpha$ turns, the quadrature-axis winding has $a \sin \alpha$ turns. However, *the two windings have to be assumed to be connected in series* (Fig. 5.13*c*) to have the same effect as the original coil. The cross-field equivalent circuit of the two windings *before* interconnection is shown in Fig. 5.13*b*. The effect of series connection is to make $i^{qs} = i^{ds}$. Hence, if the common branch is opened up (Fig. 5.13*c*), the resultant large mesh represents the equivalent circuit of the coil at an angle. It is important to remember that the design constants appear-

ing in the equivalent circuit have a counterpart in the actual machine only *after* the interconnection. Hence the design constants are known only for Fig. 5.13*c* and not for Fig. 5.13*b*.

Any other winding (or windings) on the stator must be assumed to belong to a different layer. In the presence of an additional winding along the direct axis, the resultant equivalent circuit is given in Fig. 5.14.




(*a*) Actual winding (*b*) Three equivalent meshes (*c*) Reactance definitions

FIG. 5.14. Two stator windings, one at an angle.

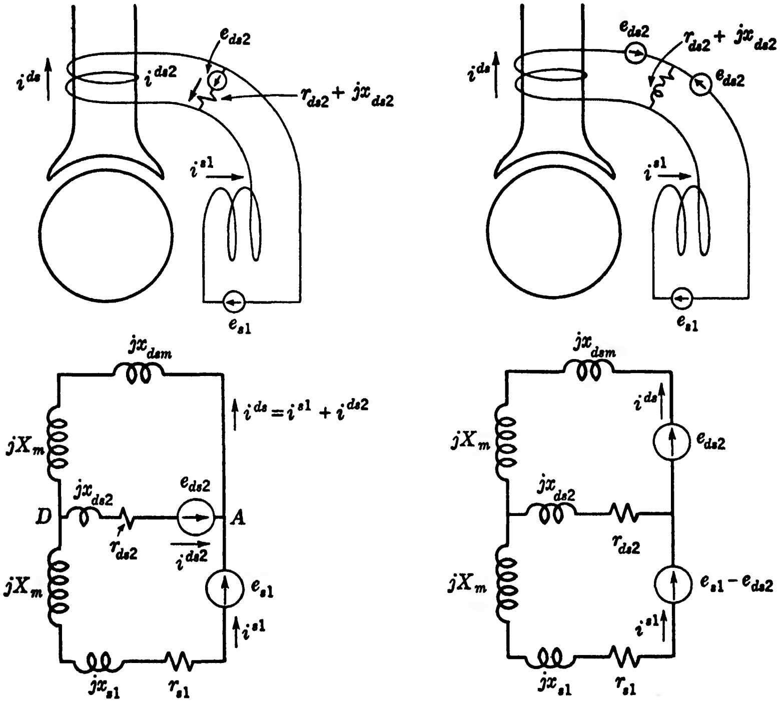
In removing the common branch of the coil at an angle, the common branch of the other coil is left undisturbed.

## REPLACING CURRENTS BY MMF'S

The equivalent circuit of Fig. 5.14*c* (and the equations describing it) is brought to a more familiar form simply by rearranging the circuit so that the *ABCD* branch in Fig. 5.14*c* becomes a common branch in Fig. 5.15*a*. Because of the rearrangement the old variable $i^{ds2}$ is replaced by the new variable $i^{ds}$, where $i^{ds} = i^{s1} + i^{ds2}$. The impressed voltage in the common branch $e_{ds2}$ may be shifted into the two outside branches as shown in Fig. 5.15*b*.

Now the stator equivalent circuit looks the same as if the two windings were at right angles in space, with the difference that now a mutual impedance also exists between the two windings. The new variable $i^{ds}$ is the resultant mmf acting in the direct axis, and $i^{s1}$ is the resultant mmf acting in the quadrature axis.

Hence *two windings placed on the stator at an angle $\alpha$ different from* 90° *may always be replaced with two windings at right angles, having, however, a mutual impedance between them. The currents in the windings at right angles represent the mmf's of the windings at an angle in the two rectangular directions* $\mathbf{d}_s$ and $\mathbf{q}_s$.




(*a*) Fig 5.14c rearranged (*b*) $e_{ds2}$ shifted out

FIG. 5.15. Replacing currents by mmf's.

It should be expressly noted that during all these changes the airgap reactances, $x_{md}$ and $x_{mq}$, remain undisturbed. Hence no change is introduced in the rotor meshes at standstill or at any speed.

## TWO-PHASE IMPEDANCES WITH MUTUAL COUPLING

Two stator windings at an angle are analogous to the problem of adding to the stator of a standard induction motor two unbalanced impedances $Z_d$ and $Z_q$ that differ from each other and have a mutual impedance $z_m$ between them.

The equivalent circuit of such a two-phase impedance along the physical $\mathbf{d}$ and $\mathbf{q}$ axes is shown in Fig. 5.16*a*. In passing over to sequence

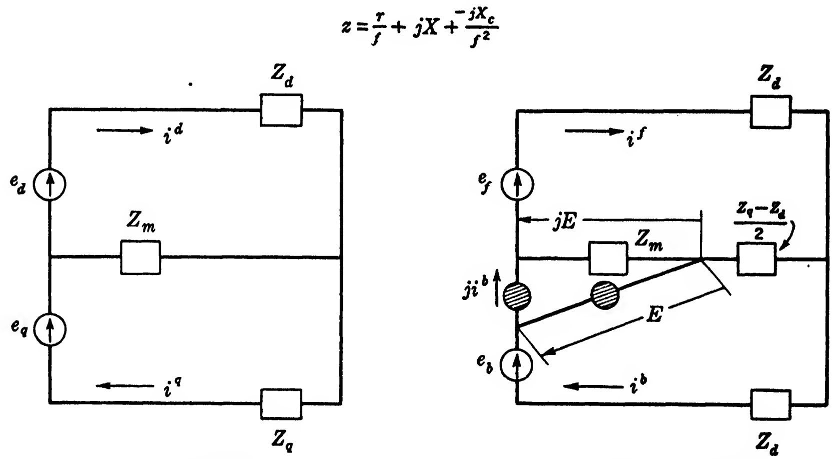
axes, the transformation of $Z_d$ and $Z_q$ has been given in Fig. 2.7. However, the manner of representation of a mutual impedance $Z_m$ still has to be determined. If the steps in Eqs. 2.3 to 2.5 are followed, the equations of Fig. 5.16*a* are (considering only $Z_m$)

$$e_d = Z_m i^q \quad \text{and} \quad e_q = Z_m i^d \tag{5.1}$$

If symmetrical components are introduced by Eq. 2.9, the new equations are

$$e_f = jZ_m i^b \quad \text{and} \quad e_b = -jZ_m i^f \tag{5.2}$$

In analogy to Fig. 3.3*b* such an asymmetrical mutual impedance is represented by a phase shifter (Fig. 5.16*b*) rotating the current $i^b$ and the voltage $E_b$ by $j$ (or 90°$f$).




(*a*) Cross field (*b*) Revolving field

FIG. 5.16. Two-phase impedances with mutual coupling along stationary axes.

## SHADED-POLE MOTOR AT STANDSTILL

In a shaded-pole motor, Fig. 5.17, a short-circuited coil (or shaded coil) embraces part of the stator salient pole in order to displace in space the stator flux from the stator current during starting and produce thereby a starting torque.

There are several ways of defining the reactances of such a structure. In Fig. 5.17, the main winding is assumed to lie along the direct axis, and the shaded coil is at an angle $\alpha$ from it. (Another assumption might have been to consider the shaded-coil axis as the quadrature axis and the main winding shifted from it by an angle.) Hence the shaded coil is assumed to be wound on both **d** and **q** axes. The reactances due

to such an assumption are shown on both Figs. 5.17 and 5.14*c*. To make the analysis applicable to induction motors also, in which the two stator windings are actually placed at an angle $\alpha$ different from 90°, or to brake motors, an impressed voltage $e_s$ is assumed on the shaded coil.

The disadvantage of this method of definition of reactances is that the angle of displacement $\alpha$ of the shaded coil and its ratio of turns $a$ do not occur explicitly in the reactances. However, *it is possible to*

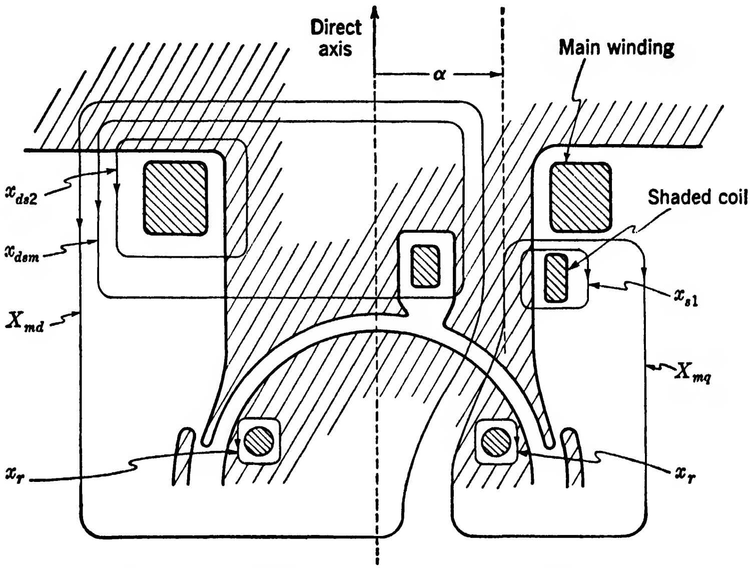



FIG. 5.17. Shaded-pole motor fluxes and reactances.

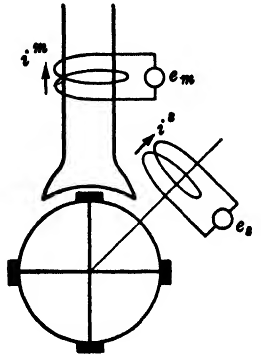
*define the given stator reactances and resistances as functions of $\alpha$ and $a$*, as is shown in Appendix 2.

The relations between the currents, voltages, and design constants of the actual shaded-pole motor and those of an equivalent split-phase motor with windings at right angles are shown in Fig. 5.18. The primed quantities are the stator design constants of the equivalent motor.

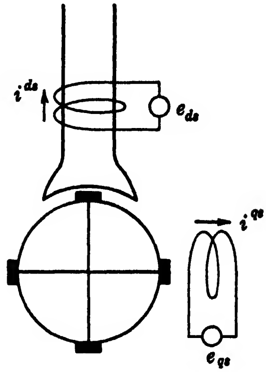
## EQUIVALENT CIRCUITS OF THE SHADED-POLE MOTOR *

Since in Figs. 5.14 and 5.15 the quadrature-axis currents are expressed as $i^q$ (and not $ji^q$), the effect of the rotation is to introduce a mutual resistance term in the rotor with a phase shifter, Fig. 5.19*a*, in accordance with the primitive circuit of Fig. 3.5*b*. The impressed

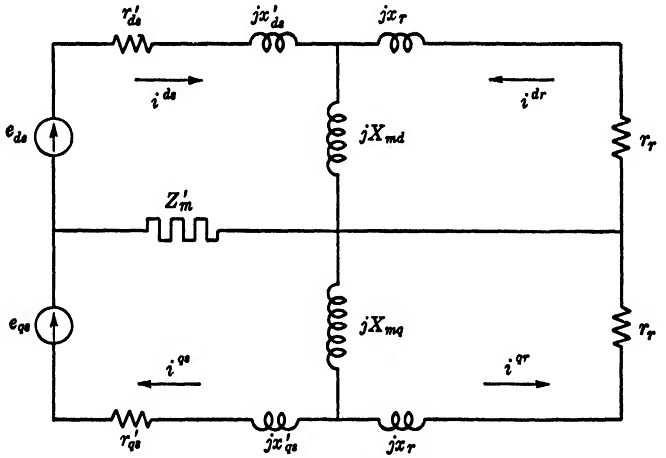
* Gabriel Kron, "Equivalent Circuits of the Shaded-Pole Motor with Space-Harmonics," *Transactions of the AIEE*, Vol. 69, pp. 735–41, 1950.




(*a*) Shaded-pole motor




(*b*) Equivalent motor



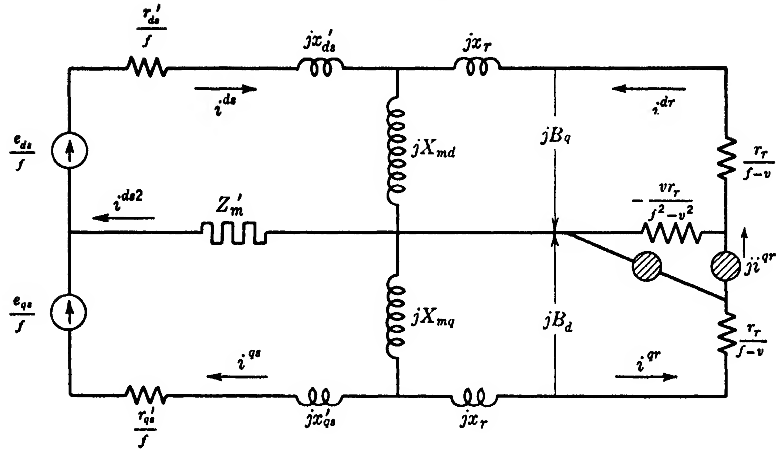

(*c*) Reactances of the equivalent motor

| | |
|---|---|
| $i^m = i^{ds} - i^{qs}$ | $e_{ds} = e_m$ |
| $i^s = i^{qs}$ | $e_{qs} = e_s - e_m$ |

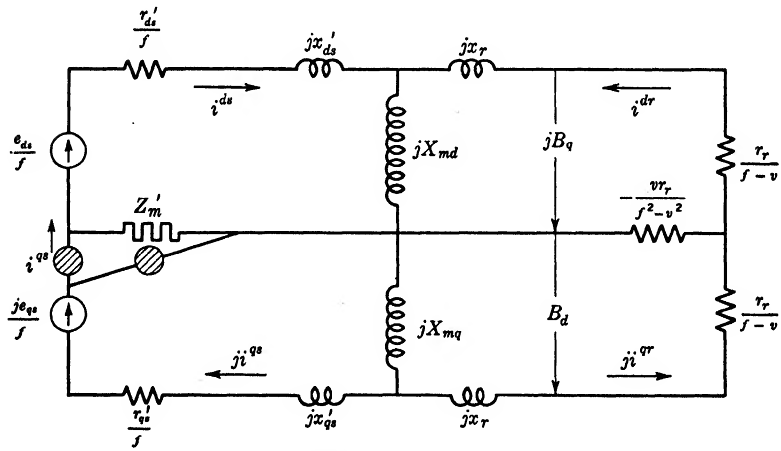
| | | |
|---|---|---|
| $r'_{ds} = 0$ | $r'_{qs} = r_{s1}$ | $z'_m = r_{ds2} + jx_{ds2}$ |
| $x'_{ds} = x_{dsm}$ | $x'_{qs} = x_{s1}$ | |

(*d*) Relations between shaded-pole and equivalent motor

Fig. 5.18. Equivalent split-phase motor at standstill (cross-field network).




(a) Phase shifter in rotor




(b) Phase shifter in stator

FIG. 5.19. The shaded-pole motor (cross-field theory).

voltage $e_{qs}$ stands for $e_s - e_m$, and $e_{ds}$ for $e_m$. In general, *the final equivalent circuits of the shaded-pole motor, Fig. 5.19, are given in a form that is valid for any type of definition of the reactances,* not only that of Fig. 5.16*a* (another definition of the reactances of the equivalent split-phase motor is given in Appendix 4).

The phase shifter may be eliminated from the rotor (according to the principles of Chapter 3), by replacing all $i^q$ by $ji^q$ and $e_q$ by $je_q$. (However, because of the common branch in the stator, which acts as a barrier, one end of the phase shifter remains fixed in the network, [Fig. 5.19*b*].)

The circuit with $ji^q$ may be transformed to sequence axes, resulting in the circuit of Fig. 5.20. Since the circuit of Fig. 5.19*b* contains a mutual

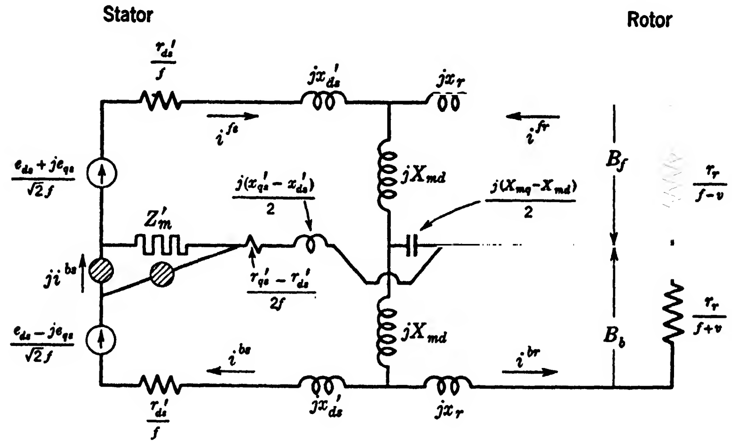



FIG. 5.20. The shaded-pole motor (revolving-field theory).

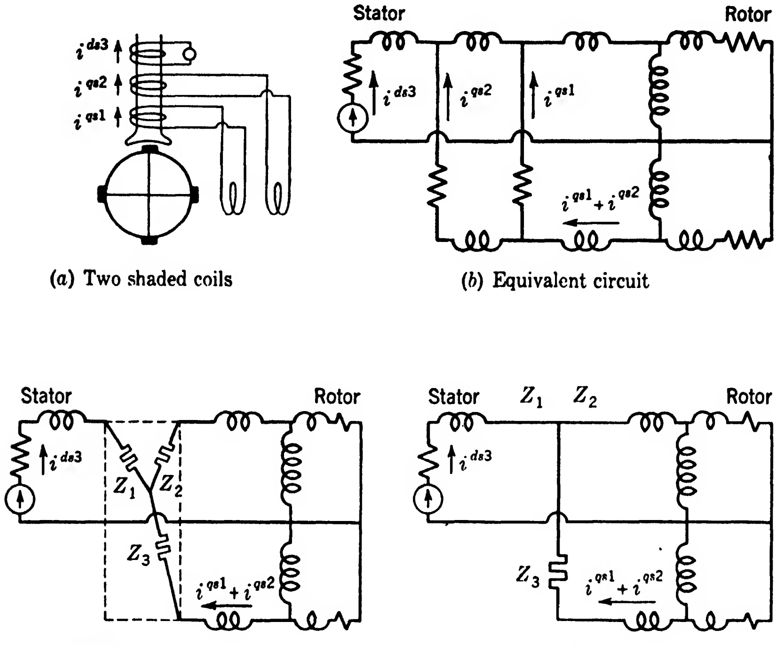
branch in the stator, the same branch appears after transformation also with a phase shifter. *No reference frame seems to be available that makes the phase shifter unnecessary.*

Since the given analysis of a shaded-pole motor is often inadequate if the effect of the space-harmonic fluxes is ignored, the given equivalent circuits are extended to include the effect of space harmonics in Chapter 10.

## A SECOND SHADED COIL

An additional shaded coil may again be represented by a second set of direct- and quadrature-axis coils connected in series (Fig. 5.21*a*). Their equivalent circuit at rest, analogous to Fig. 5.14*c*, is shown in Fig. 5.21*b*.

The four impedances of the second stator mesh (containing $i^{qs2}$) may be changed by a mesh-star transformation into Fig. 5.21*c* with three impedances. Arranging the three impedances as shown in Fig. 5.21*d*, the original Fig. 5.14*c* is reestablished, except that the stator resistance $r$ and inductances $jx$ each become now more general impedances $z$.




(*a*) Two shaded coils (*b*) Equivalent circuit

(*c*) Mesh-star transformation (*d*) Equivalent single shaded coil

FIG. 5.21. Elimination of second shaded coil.

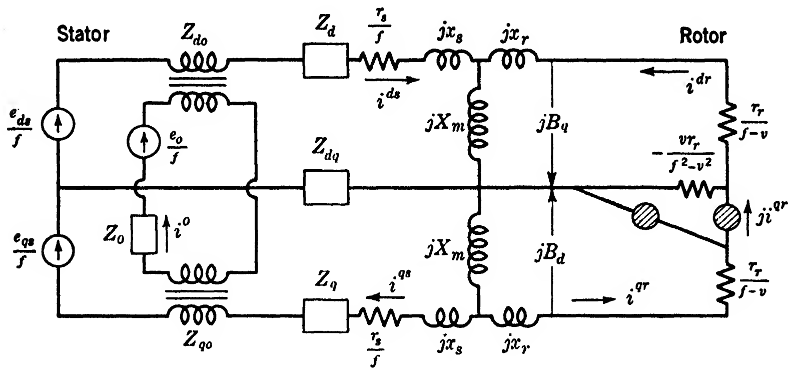
## BRAKE MOTORS

Often the windings of a *standard three-phase* induction motor are interconnected among themselves and with outside impedances in order to produce desired speed-torque characteristics. All such connections may be reduced to that of an *unbalanced two-phase* induction motor coupled to a zero-sequence mesh.

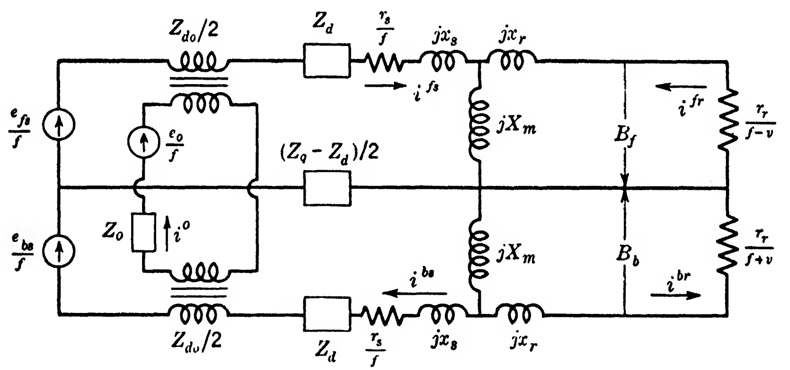
When all three phases are unequal, then only the cross-field network of Fig. 5.22*a* can be established for the system. With the phase shifter in the rotor, all mutual couplings between the stator $d$, $q$, and 0 axes are reciprocal. In the revolving-field networks the couplings would be non-reciprocal.

If, however, the stator connections are symmetrical with respect to one of the phases, as is the case in most brake motor connections, then

the revolving-field network (Fig. 5.22*b*) also has only reciprocal mutuals between the **f**, **b**, and **0** axes.




(*a*) Cross-field network
(all stator self-impedances and mutual impedances unequal)




(*b*) Revolving-field network
(stator self-impedances and mutual impedances symmetrical with respect to one phase)

FIG. 5.22. Brake motor. (Unbalanced *three-phase* induction motor.)

## THREE-PHASE INDUCTION MOTORS WITH SPECIAL WINDINGS

When the stator windings of a three-phase induction motor have also an unequal number of turns or have unusual double-circuit connections, the equivalent circuits of Fig. 5.22 still apply for them.

If, however, some of the turns are left unexcited, so that the 120° spacing of the phases is disturbed, the equivalent circuits have to be extended.

# 6 SYNCHRONOUS MACHINES

## AMORTISSEUR WINDINGS

Synchronous machines operating below or at synchronous speed also have two types of equivalent circuits, namely, the "cross-field" circuits along the physical **d** and **q** axes and the "revolving-field" circuits along the hypothetical **f** and **b** axes. However, synchronous machines differ from induction machines not only in the value of impressed frequencies but also in the manner of impressing voltages. Some of their windings occasionally need special considerations.

Since the amortisseur windings located in the field structure are permanently short-circuited, it is customary to reflect all the field and amortisseur meshes that exist on the salient structure into the armature meshes on the smooth structure. The armature meshes contain now, however, short-circuit (or operational) impedances and impressed voltages. Hence, *in general, two sets of equivalent circuits (with and without the field meshes) will be given* for each type of operation of a synchronous machine. The resultant circuits may also be simplified still further, as will be shown presently, thereby increasing the possible number of equivalent circuits to at least five types and often to more.

In many problems the currents in the *individual bars* of the amortisseur have to be known. For such cases more complicated equivalent circuits must be developed, showing the reactances and resistances of each amortisseur bar and the end rings connecting them. The bars may be identical and uniformly spaced, or they may be dissimilar and non-uniformly distributed around the periphery of the field structure. These more general circuits are considered in Chapter 8.

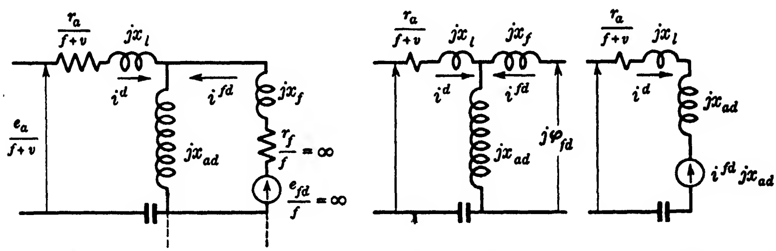
*Solid rotors* are usually represented by several concentric standard windings to simulate the effects of eddy currents in the iron. It will be shown in Chapter 8 that the circular cross-section of a solid rotor may be more accurately represented by a two-dimensional network that satisfies the field equations of Maxwell.

Although the armature usually has one layer of winding, occasionally special machines are built with several armature layers. One such

machine, the "slip coupling," will also be analyzed. Also a double-winding generator has to be considered as a machine with two separate layers of winding under unbalanced conditions (such as a turn-to-turn short circuit), as will be shown in Chapter 12.

## FREQUENCY OF IMPRESSED VOLTAGES

On a synchronous machine the voltages impressed or induced at the *armature* terminals are measured not along the physical **d** and **q** axes (that rotate with the field) but along axes stationary in space. For




(*a*) Impressed field voltage (*b*) Impressed field current (*c*) Impressed armature voltage

FIG. 6.1. Representation of a d-c field voltage ($f = 0$).

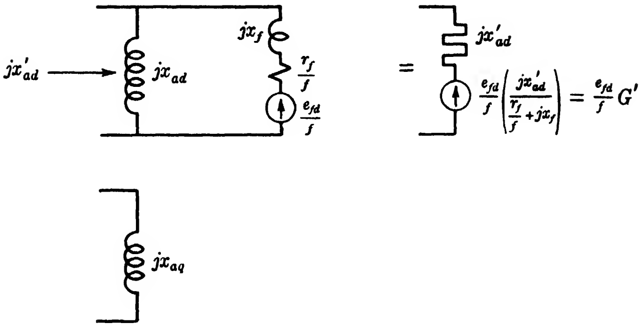
purposes of analysis these impressed voltages must be referred to the **d** and **q** axes. For instance, if the frequency of a positive-sequence terminal voltage is unity along the stationary terminals, its frequency becomes $1 - v = s$ (or slip frequency) along the physical axes. Hence $f$ in the primitive network becomes $f = s = 1 - v$.

On the other hand, the frequency of the impressed voltage on the *field* is zero ($f = 0$). The field resistance in Fig. 6.1*a* is $r_f/f = \infty$, and the field winding is open-circuited. However, the impressed voltage $e_{fd}/f$ in the equivalent circuit is also infinite and the ratio (representing the field current $i^{fd}$) is finite, namely, the actual d-c field current. In consequence, *the d-c current is represented by a known current source* $i^{fd}$ (Fig. 6.1*b*).

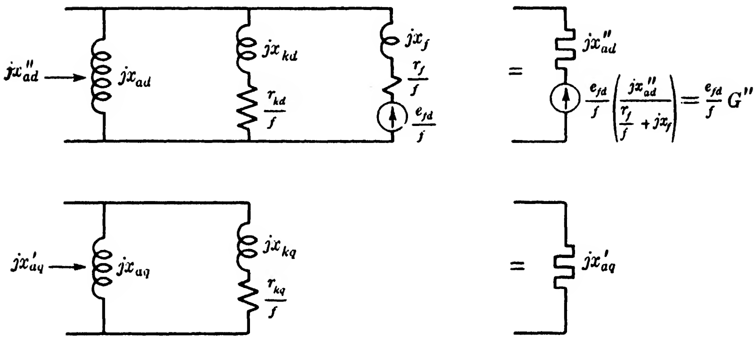
If the field leakage reactance $jx_f$ is included in the equivalent circuit, the voltage across it represents the voltage induced $j\phi_{fd} = jB_{fq}$ by the resultant flux $\phi_{fd}$ linking the field winding. A value of this flux is often needed in transient studies and in voltage regulator applications.

## SHORT CIRCUIT OR OPERATIONAL IMPEDANCES

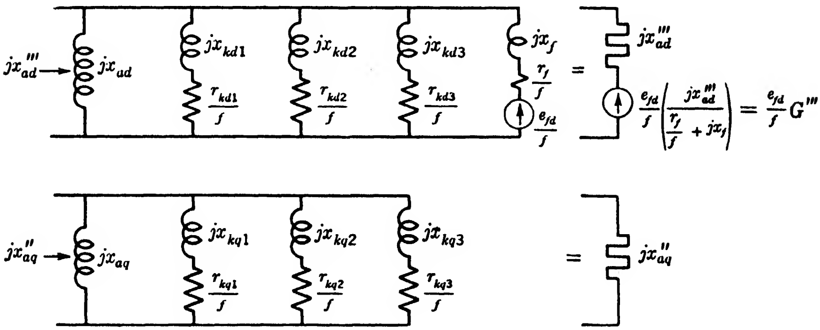
No matter how complicated the amortisseur winding representation is, the whole field structure and the airgap in each axis may be replaced




(*a*) No amortisseur




(*b*) Standard amortisseur




(*c*) Solid rotor

FIG. 6.2. Representation of field and amortisseur windings.

by a single equivalent impedance, denoted by $jx''_{ad}$ along the direct axis (Fig. 6.2*b*) and by $jx'_{aq}$ along the quadrature axis. (Usually each mesh considered adds one prime to the symbol; hence the extra field winding on the direct axis adds an extra prime to $x'_{ad}$.) Both impedances are measured as viewed from the armature meshes and include the airgap reactances in their definition. If the armature leakage reactances, $x_l$ (equal in both axes), are also included in the definition, they are denoted by $jx''_d$ and $jx'_q$. The effect of rotation of the field structure is taken care of in these operational impedances by dividing all resistances by the absolute frequency $s$ = slip $= f - v$.

*The elimination of the field meshes in the equivalent circuits consists merely in replacing in the armature meshes the airgap reactances* $x_{ad}$ *by* $x''_{ad}$ *and* $x_{aq}$ *by* $x'_{aq}$. These substitutions are valid for all cross-field and revolving-field equivalent circuits.

Whereas an amortisseur winding is usually represented by a single layer of winding ($x''_{ad}$ and $x'_{aq}$) (Fig. 6.2*b*), a solid rotor is represented by two or more windings in parallel (Fig. 6.2*c*) ($x'''_{ad}$ and $x''_{aq}$). When the amortisseur is absent or the current in it is zero (Fig. 6.2*a*), the impedances are denoted by $x'_{ad}$ and $x_{aq}$. When a field winding also exists along the quadrature axis, $x_{aq}$ is replaced by $x'_{aq}$ or $x''_{aq}$.

## THE TRANSFERRED IMPRESSED VOLTAGES

Whenever a mesh is eliminated, any impressed voltage or impressed current existing in the mesh is transferred into several of the remaining meshes after being multiplied by a factor $G$. In particular:

1. A field impressed *current* $i^{fd}$ appears in the armature as an impressed voltage $i^f jx_{ad}$ in series with $jx_{ad}$ after elimination of the field mesh (Fig. 6.1*c*). (In the original equivalent circuit [Fig. 6.1*b*] the effect of $i^{fd}$ is to induce the voltage $i^{fd}jx_{ad}$ in its neighboring mesh.)

2. A field impressed *voltage* $(e_{fd}/f)$ appears in the armature mesh as an impressed voltage $(e_{fd}/f)\,G$ in series with the impedance $jx''_{ad}$, as shown in Fig. 6.2*b*. The value of $G$ is calculated by writing three voltage equations for the three direct-axis meshes of Fig. 6.2*b* and eliminating the two field currents. The result is $G = jx''_{ad}/(r_f/f + jx_f)$.

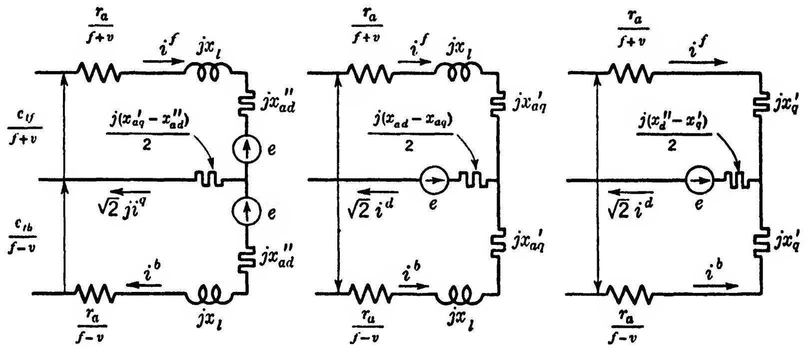
It should be emphasized that *the process of elimination of field meshes cannot be considered a "transformation" of reference frames.* (Nor can it be considered, as some authors regard it, a new type of "per-unit" system.) *Dynamically, the process corresponds to "ignoration of coordinates."*

Hence, in addition to the complete cross-field and revolving-field circuits, in which all the field meshes are identified, another set of

equivalent circuits exist for synchronous machines, in which all field meshes are reflected back to the armature meshes.

## FIELD VOLTAGE IN THE SEQUENCE NETWORK

If the field meshes are eliminated in the revolving-field networks, the impressed field voltage (or current) appears in both **f** and **b** meshes, Fig. 6.3*a*. (In the cross-field equivalent circuits [Fig. 6.1] $e_{fd}$ appears only in the **d** mesh.) Since the two impressed field voltages are equal, they may be replaced by a single field voltage in the common branch.



(*a*) Two field voltages (*b*) One field voltage (*c*) $x_q$ combined with $x_{aq}$

FIG. 6.3. Field voltage in the armature (revolving-field network).

As a result, *the two armature voltages and the field impressed voltage are measured in the network from a common ground* (*Fig. 6.3b*). This development can be accomplished only if the sign convention of the primitive machine of Fig. 2.8*b* is used. The steps are as follows (Fig. 6.3):

1. The direction of $i^b$ of Fig. 6.3*a* is reversed in Fig. 6.3*b*.
2. $x''_{ad}$ in the vertical branches is replaced by $x'_{aq}$.
3. The sign of the impedance in the mutual branch (representing the saliency) changes.

With this simplification the number of possible equivalent circuits of a synchronous machine becomes five.

It should be noted that after eliminating the field meshes, the leakage reactance $x_l$ may be combined with $x''_{ad}$ and $x'_{aq}$ to give $x''_d$ and $x'_q$, as shown in Fig. 6.3*c*. The saliency reactance $x''_{ad} - x'_{aq}$ may be replaced by $x''_d - x'_q$ since $x_l$ cancels.

## IGNORATION OF THE BACKWARD MESHES

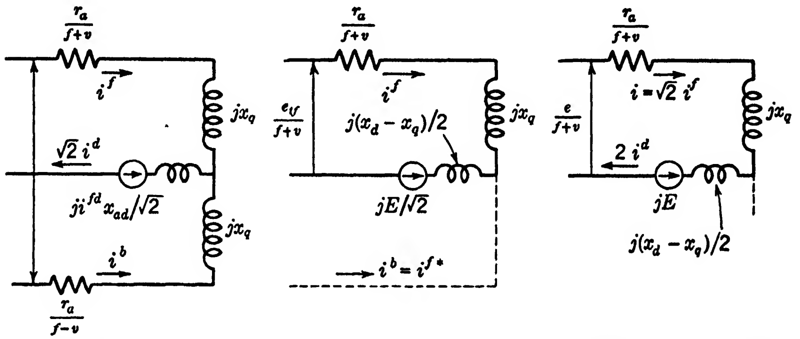
When a synchronous machine operates under such conditions that no currents flow in the amortisseur bars ($f = 0$) a further simplification

may be introduced. If (and only if) the time phase is so selected that the *field current* $i^{fd}$ *and field impressed voltage* $e_{fd}$ *are real numbers,* then the currents in the backward meshes are the conjugates of the currents in the forward meshes. In particular:

$$\begin{array}{ll|ll} i^b = i^{f*} & & e_b = -e^*_f & \\ B_b = B^*_f & & Z_b = -Z^*_f & \end{array} \qquad 6.1$$

Thereby, for calculation purposes only, *the backward meshes may be left out* (Fig. 6.4).

Since $\sqrt{2}i^d$ and not $i^f$ flows in the common branch, the resulting equivalent circuit—containing **f** meshes only—is not a self-consistent




(*a*) Both meshes (*b*) Forward mesh only (*c*) Multiplying *b* by $\sqrt{2}$

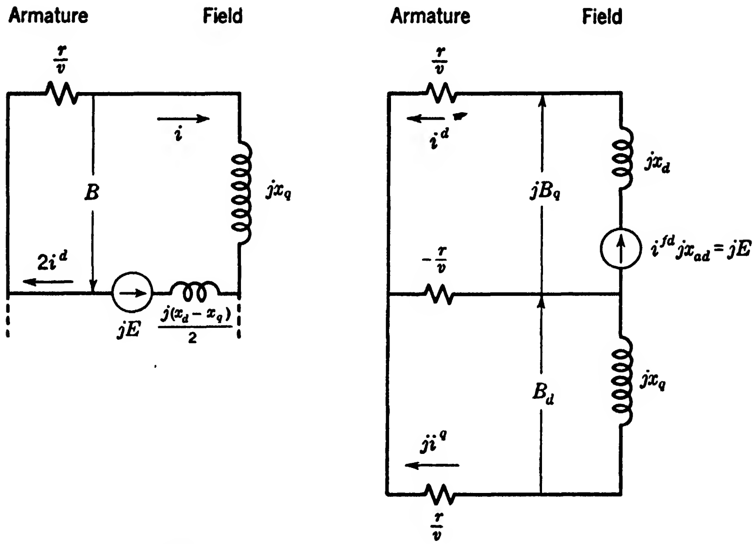
FIG. 6.4. Ignoration of backward meshes ($i^b = i^{f*}$).

network to be used on an a-c network analyzer. By various tricks to be shown below, the simplified circuit may, however, be made practical for use on the a-c analyzer.

Since in the forward mesh $i^f = (i^d + ji^q)/\sqrt{2})$ *it is possible to multiply all currents and voltages in the forward mesh by* $\sqrt{2}$ as shown in Fig. 6.4*c*, thereby replacing $i^f$ by the more conventional resultant armature current $i = i^d + ji^q = \sqrt{2}i^f$.

## SYNCHRONOUS MACHINE EXCITED ON THE FIELD ONLY

Since the frequency of the impressed field voltage is zero, it has been shown in Fig. 6.1 that $e_{fd}$ is replaced by an impressed field current $i^{fd}$ and that $i^{fd}$ is replaced by an impressed armature voltage $i^{fd}jx_{ad}$. Also the amortisseur meshes are open-circuited during steady state. The two types of networks are shown in Fig. 6.5, assuming the velocity $v$ of the field structure a variable quantity. At synchronous speed, $v = 1$; below synchronous speed, $v$ is less than unity.

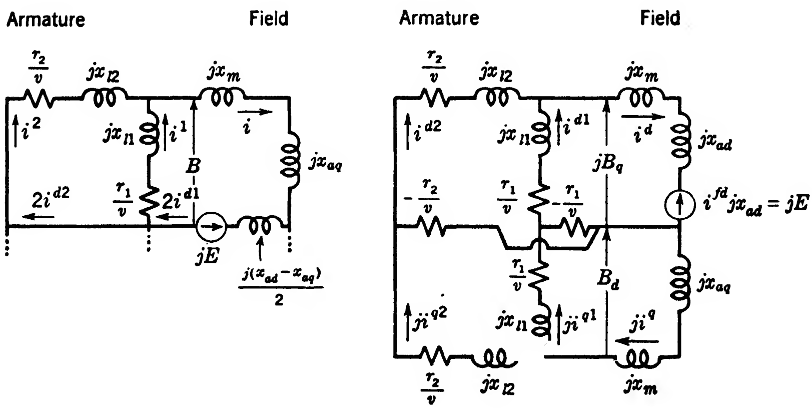



(a) Revolving field (b) Cross field

FIG. 6.5. Synchronous machine excited on the field only (field mesh eliminated).

## SLIP COUPLING

Synchronous machines have been built (the so-called "slip couplings") with a double squirrel-cage winding on the armature, in addition to the amortisseur on the field (Fig. 6.6). The field was excited by a d-c source.




(a) Revolving field (b) Cross field

FIG. 6.6. Slip coupling.

## SYNCHRONOUS MACHINE EXCITED ON THE ARMATURE ONLY

A synchronous machine is started up by exciting the stationary axes of the armature by an emf rotating at unit speed in the direction of the rotor speed $v$. Across the **d** and **q** armature reference frame (rotating at a velocity $v$) the impressed emf appears rotating forward with slip frequency $s = f - v$. Hence, $f$ in the primitive machine becomes $s$. The various forms of the equivalent circuits are shown in Fig. 6.7*a* and *b* and in Fig. 6.8*a* and *b*. The backward rotating impressed voltage $e_b$ is assumed to be zero.

At half synchronous speed, $s = v$ and $s - v = 0$. The corresponding circuits are shown in Fig. 6.9. At synchronous speed it is a so-called "reaction" machine, Fig. 6.10, and the rotor runs in synchronism with the revolving field only because of its saliency.

Occasionally the problem of *an armature excited by an unbalanced voltage* arises. Then a backward rotating field, $e_b$, also exists. Across the **d**, **q** axes the frequency becomes $f + v$. Hence, another set of equivalent circuits have to be established (Fig. 6.7*c* and *d* and Fig. 6.8*c* and *d*) in which only $e_b$ is impressed and in which $f$ is replaced by $f + v$.

Another point of view towards these circuits is illustrated in Fig. 11.5.

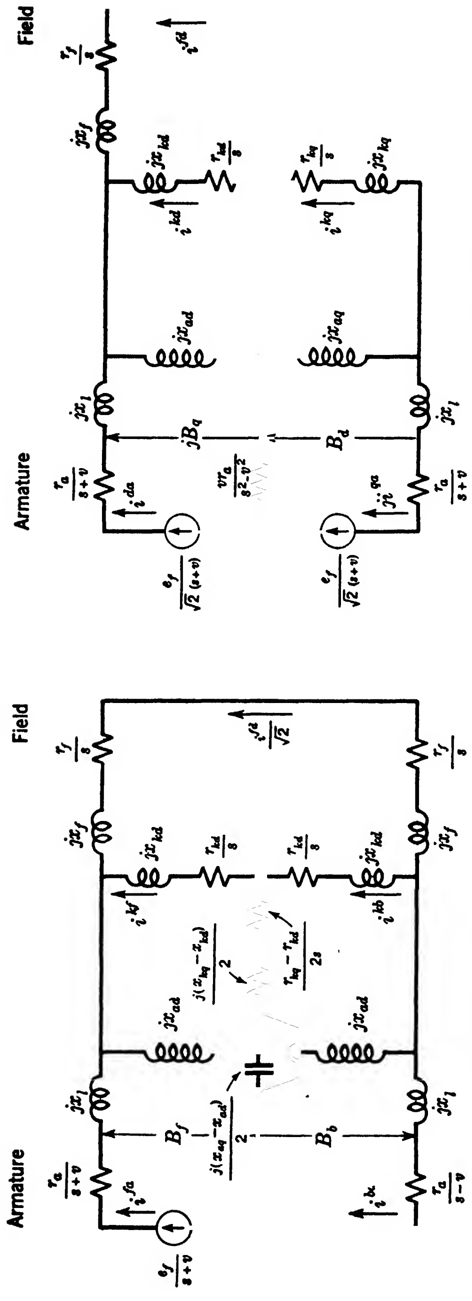
## SYNCHRONOUS MACHINE EXCITED ON BOTH FIELD AND ARMATURE OPERATION DURING STARTING *

When both field and armature are excited (the armature only by a forward rotating voltage) and the rotor runs below synchronous speed, each impressed voltage produces currents of different frequencies in all windings along the **d**, **q** axes. Hence, a different equivalent circuit is associated with each impressed voltage. These circuits were already shown individually and are shown again in Fig. 6.11. One of the major effects of the coexistence of the two sets of currents in the same machine is to introduce several torques with variable frequencies. Since the denominators of the armature resistances indicate the current or flux frequencies, their sums or differences indicate the torque frequencies.

## CALCULATION OF ALTERNATING TORQUES DURING THE STARTING PERIOD

The torque calculations will be shown only for the revolving-field networks. It will be assumed also that only a positive-sequence (forward) voltage is impressed on the armature. The frequencies of the

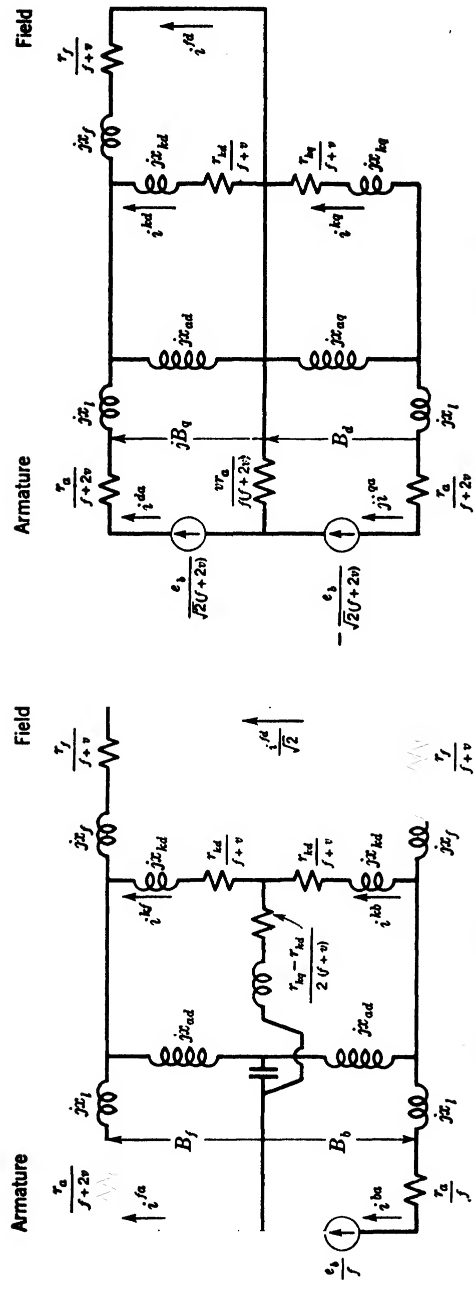
* Gabriel Kron, "Equivalent Circuit of the Salient-Pole Synchronous Machine," *General Electric Review*, Vol. 44, p. 679, December 1941.




(a) Revolving field

(b) Cross field

(A) With forward rotating emf applied ($f$ becomes $f - v = s$)

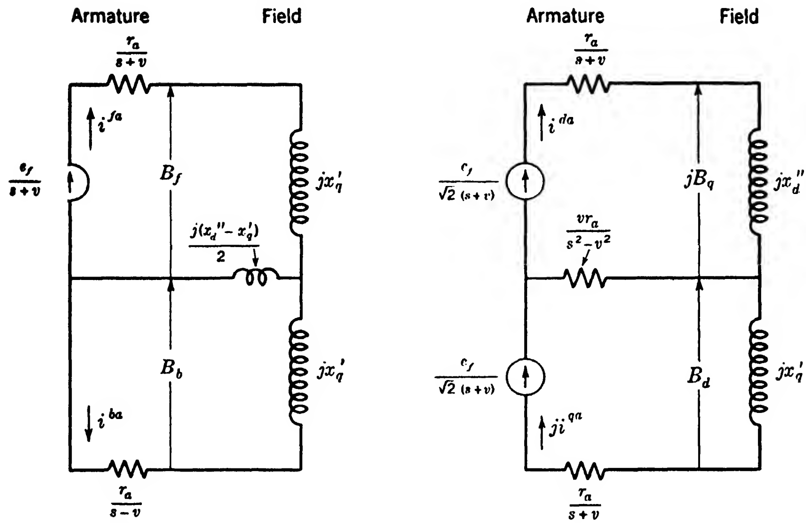



(c) Revolving field

(d) Cross field

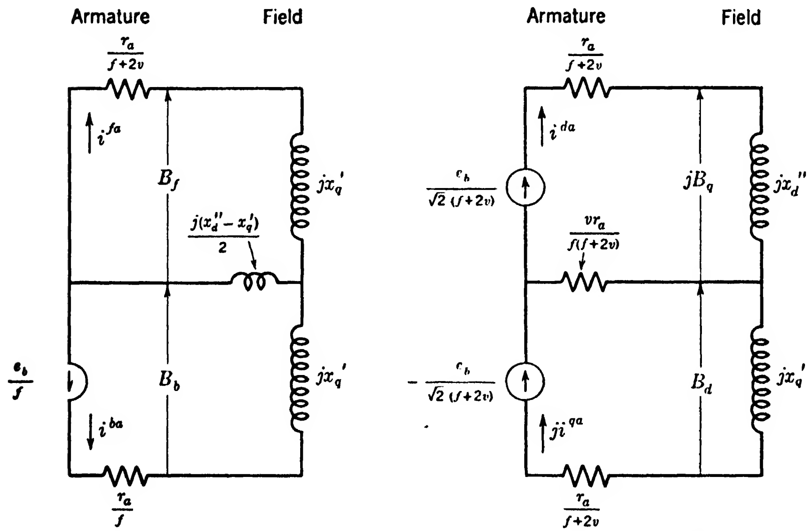
(B) With backward rotating emf applied ($f$ becomes $f+v$)

FIG. 6.7. Synchronous machine excited on the armature only.




(*a*) Revolving field (*b*) Cross field

(*A*) With forward rotating emf applied ($f$ becomes $f - v = s$)




(*c*) Revolving field (*d*) Cross field

(*B*) With backward rotating emf applied ($f$ becomes $f + v$)

FIG. 6.8. Synchronous machine excited on the armature only (field meshes eliminated).

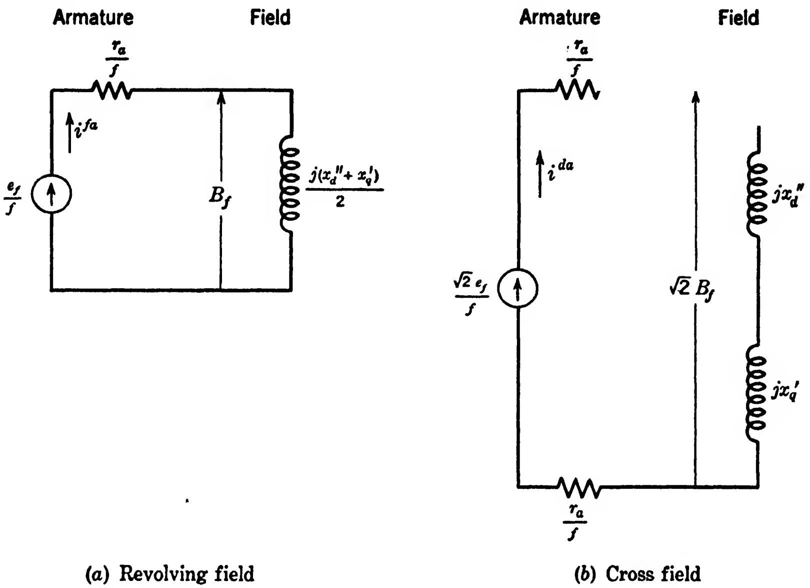



FIG. 6.9. Synchronous machine running at half speed ($v = f/2$) (forward excitation on armature only).

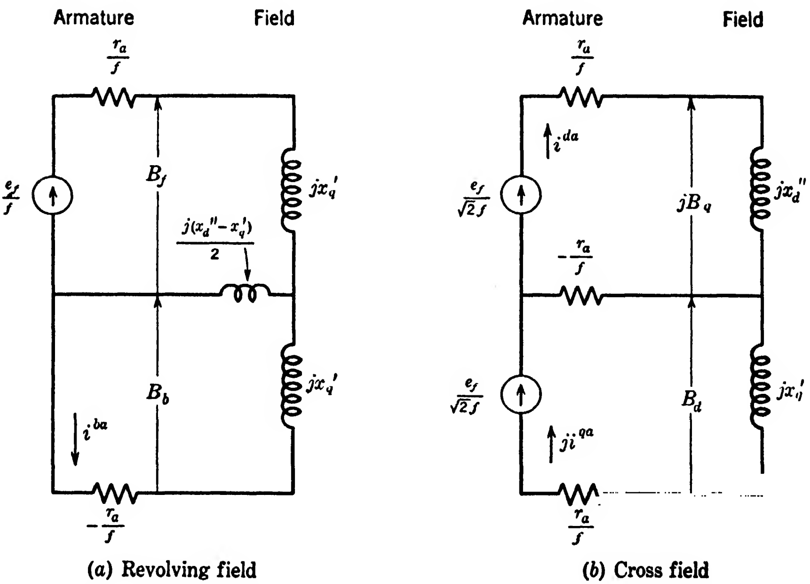
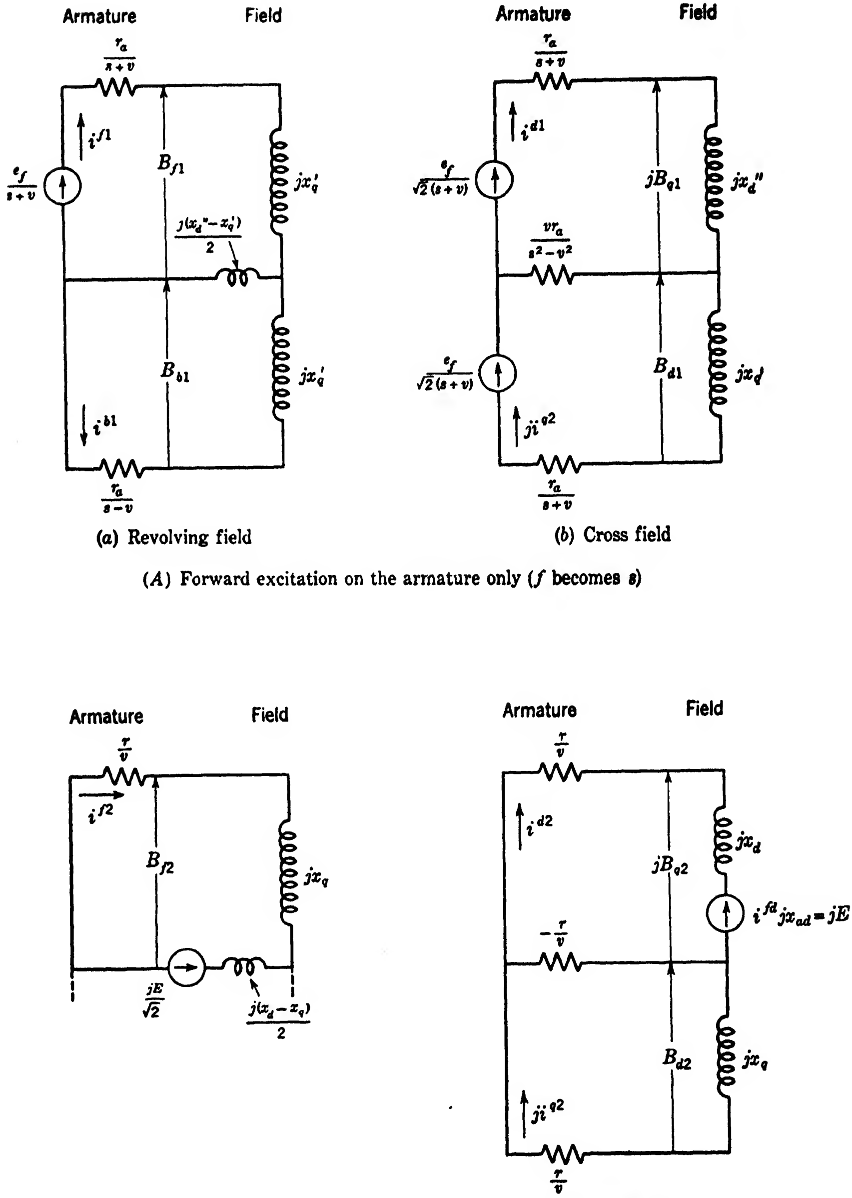



FIG. 6.10. Reaction machine ($v=f$) (forward excitation on armature only).




(a) Revolving field

(b) Cross field

(A) Forward excitation on the armature only ($f$ becomes $s$)

Revolving field

(d) Cross field

(B) Excitation on field only ($f = 0$)

FIG. 6.11. Synchronous machine excited on both sides. Operation during starting.

coexisting currents and fluxes of Fig. 6.11*a* and *c* are (where $f$ in the primitive network has been replaced by $f - v = s = \text{slip}$)

$$i^{f1}, B_{f1} = s + v \qquad i^{f2}, B_{f2} = v$$

$$i^{b1}, B_{b1} = s - v \qquad i^{b2}, B_{b2} = -v$$

In Fig. 6.11*a* the value of $B_{b2} = B^*_{f2}$, by Eq. 6.1.

The sum-frequency torques are, by Eq. 4.7,

$$T_+ = i^f B_b + i^b B_f = (i^{f1} + i^{f2})(B_{b1} + B_{b2}) + (i^{b1} + i^{b2})(B_{f1} + B_{f2})$$

$$= i^{f1}B_{b1} + i^{f1}B_{b2} + i^{f2}B_{b1} + i^{f2}B_{b2} + i^{b1}B_{f1} + \cdots$$

The frequencies of each of these torque components are found by adding the frequencies of a current and a flux.

$$\text{Frequency of } T_+ = [(s + v) + (s - v)], [(s + v) - v], [(v + (s - v)], \cdots$$

$$= 2s, s, s, 0, 2s, s, s, 0, \cdots$$

The difference-frequency torques are, by Eq. 4.8,

$$T_- = i^{f*}B_f + i^{b*}B_b = (i^{f1} + i^{f2})^*(B_{f1}B_{f2}) + (i^{b1} + i^{b2})^* + (B_{b1} + B_{b2})$$

$$= i^{f1*}B_{f1} + i^{f1*}B_{f2} + i^{f2*}B_{f1} + \cdots$$

$$\text{Frequency of } T_- = [(-s - v) + (s + v)], [(-s - v) + v)],$$

$$[(-v + (s + v)], \cdots$$

$$= 0, -s, s, 0, 0, -s, s, 0, \cdots$$

Combining the torque components having the same frequency,

1. The *constant* (0) torques are

$$T_0 = \text{Real of } (i^{f2}B_{b2} + i^{b2}B_{f2} + i^{f1*}B_{f1} + i^{f2*}B_{f2} + i^{b1*}B_{b1} + i^{b2*}B_{b2})$$

2. The *slip-frequency* ($s$) torques are

$$T_s = (i^{f1}B_{b2} + i^{f2}B_{b1} + i^{b1}B_{f2} + i^{b2}B_{f1})$$

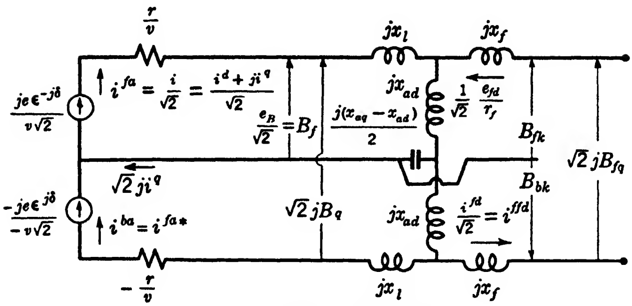
$$+ (i^{f1}B_{f2*} + i^{f2*}B_{f1} + i^{b1}{}_{b2*} + i^{b2*}B_{b1})$$

3. The double slip-frequency ($2s$) torques are

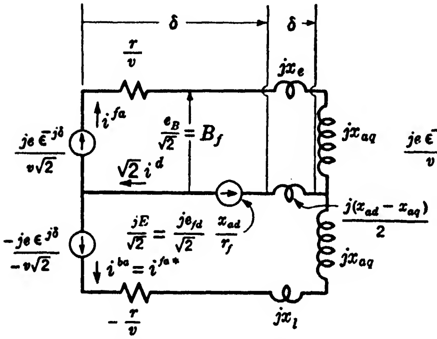
$$T_{2s} = i^{f1}B_{b1} + i^{b1}B_{f1}$$

Each torque component comes out as a complex number. The peak values of the resultant torques are, by Eq. 4.6,

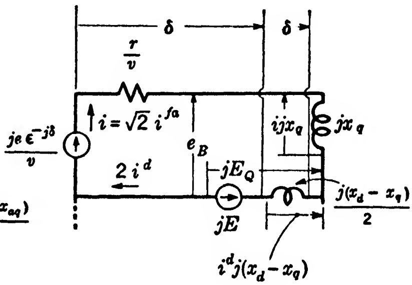
$$T_{\text{peak}} = \sqrt{\Sigma(T_{\text{real}})^2 + \Sigma(T_{\text{imag}})^2}$$



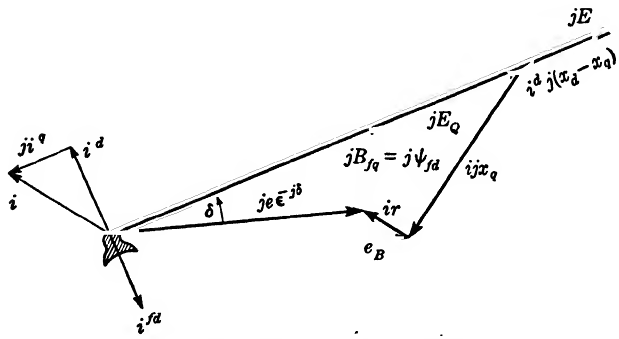

(*a*) Field mesh included




(*b*) Field mesh eliminated




(*c*) Backward mesh eliminated




(*d*) Conventional vector diagram
equivalent to circuit *c*

FIG. 6.12. Synchronous machine excited on both field and armature. Synchronous speed operation.

## SYNCHRONOUS MACHINE EXCITED ON BOTH FIELD AND ARMATURE. SYNCHRONOUS SPEED OPERATION

When both field and armature are excited and the rotor runs at synchronous speed, both applied voltages produce d-c currents along the **d** and **q** axes; hence $f = 0$. The resulting equivalent circuits are shown in Fig. 6.12.

The first circuit shows the impressed field current, the second shows its equivalent impressed voltage when it is transferred to the armature side. The transferred voltage is the so-called "voltage behind the synchronous reactance," $E = x_{ad}i^{fd}$. This voltage $E$ is also called "per-unit field current," $I_{fd}$. It is also denoted by $e_I$.

$$E = e_I = I_{fd} = i^{fd}x_{ad} = e_{fd}\left(\frac{x_{ad}}{r_f}\right) = Ge_{fd}$$

This second equivalent circuit may be solved on the a-c analyzer. If the backward mesh is eliminated and all quantities are multiplied by $\sqrt{2}$, the resultant equivalent circuit is the third circuit, Fig. 6.12*c*.

The conventional vector diagram is also included in Fig. 6.12*d* to show that the equivalent circuit of Fig. 6.12*c*, in which all field quantities have been transferred to the armature, is an exact representation of the vector diagram and concepts central station engineers constantly deal with.

## SIGN CONVENTION OF CENTRAL STATION ENGINEERS

Since this given equivalent circuit follows the *motor* sign convention (Fig. 4.2), the direction of the current vector, and hence of the induced voltages, is opposite to that occurring in the Park-Crary-Concordia convention, which holds that for one mesh

$$e_{\text{term}} = -ir - ijx + e_{\text{int}}$$

It may be mentioned that *with this sign convention the equations of a synchronous machine cannot be put into an equivalent circuit form.* This fact may have had a deciding influence on the lack of satisfactory equivalent circuits for salient-pole synchronous machines. A sign convention, or a suitable symbol, or a type of unit may appear to be a trivial matter, but often it does retard progress in certain directions for several decades.

## CONSTANT FLUX-LINKAGE NETWORKS

In many transient problems it is customary to assume that the resultant flux linking the field $\phi_{fd} = B_{fq}$ remains constant. Hence, if this constant flux linkage is assumed as an impressed voltage with zero frequency, $f = 0$, the resulting equivalent circuit of Fig. 6.13*c* is found.

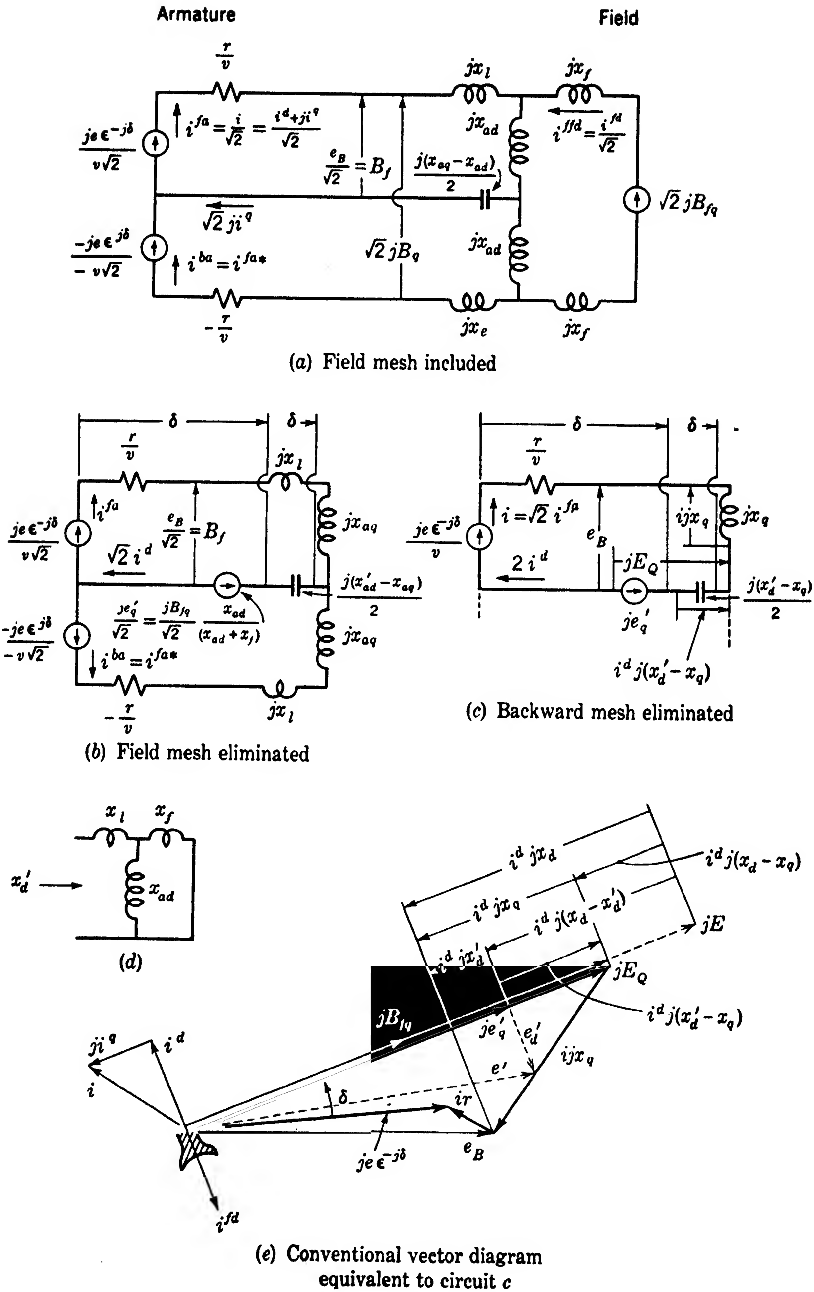



FIG. 6.13. Constant flux-linkage network.

It is the same as the steady-state network of Fig. 6.12*a*, except that the impressed current $i^{fd}$ is replaced by an impressed voltage $\sqrt{2}jB_{fq} = \sqrt{2}j\phi_{fd}$.

The impressed field voltage of Fig. 6.13*a* may be transferred to the armature, as shown in Fig. 6.13*b*. Here $x_{ad}$ of Fig. 6.13*a* is replaced by the transient reactance $x'_{ad} = x_{ad}x_f/(x_{ad} + x_f)$. The equivalent voltage $e'_q$ (shown in the common branch) is called the "voltage behind the transient reactance." If the backward mesh is ignored, the resultant circuit is shown in Fig. 6.13*c*. *This last circuit is an exact equivalent of the conventional vector diagram reproduced in Fig. 6.13e.*

In the presence of an amortisseur the same circuit is valid with the following changed interpretations:

1. $x'_{ad}$ is replaced by $x''_{ad}$ (shown in Fig. 6.2).
2. $x_{aq}$ is replaced by $x'_{aq}$.
3. $B_{fq}$ is a complex number, since $i^{fd}$ must remain a real number to be able to eliminate the backward mesh.

## FORWARD NETWORKS FOR USE ON THE A-C ANALYZER

It has been mentioned that all currents and voltages in the backward meshes of the synchronous-speed networks are the conjugates of those in the forward mesh. However, the forward mesh alone cannot be put

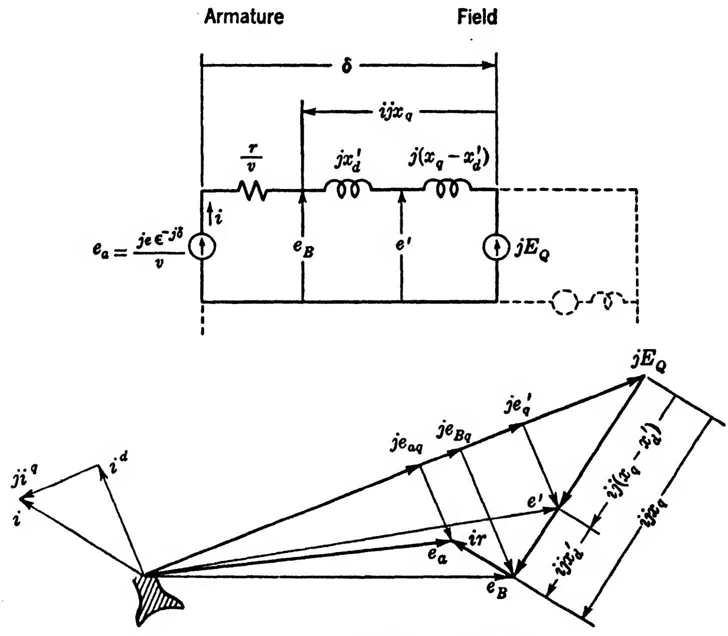



FIG. 6.14. Combined network.

on the a-c analyzer, since it is not a self-consistent mesh (in its main branches flow $i^f$; in its common branch, $2i^d$).

The forward mesh may, however, be placed on the a-c analyzer in at least two different manners:

1. The transferred field impressed voltage ($jE$ or $je'_q$) is replaced by another impressed voltage $jE_Q$ that extends from the ground to the

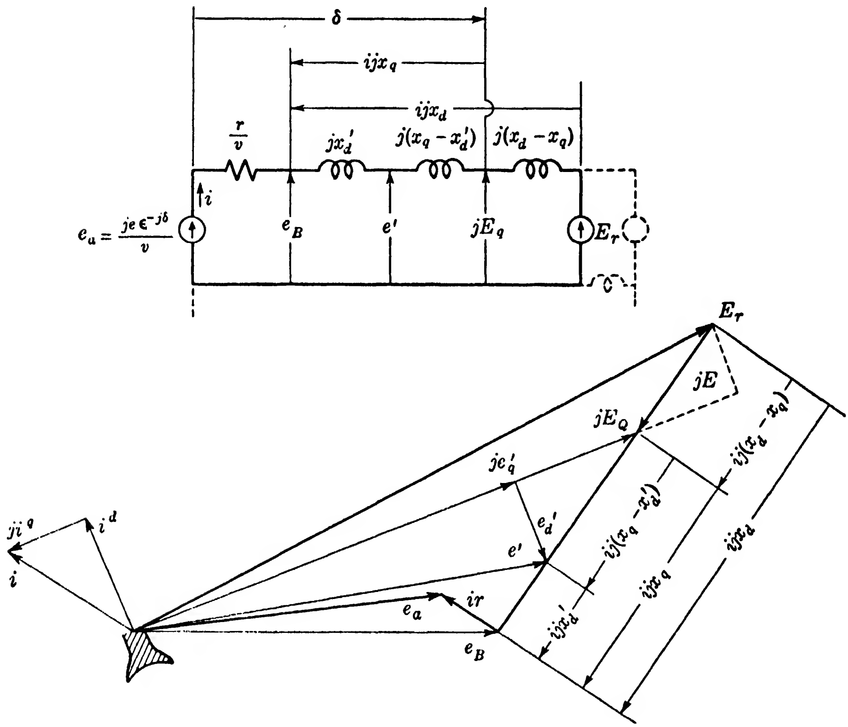



FIG. 6.15. Another combined network.

meeting points of the original **f** and **b** meshes (Fig. 6.12*c* or 6.13*c*). That is,

$$E_{\text{imp}} = jE_Q = jE + 2i^d \frac{j(x'_d - x_q)}{2}$$

$E_Q$ is called the "voltage behind the quadrature-axis reactance $x'_q$." It represents the value of $E$ (or $e'_q$) that would exist if the machine were smooth.

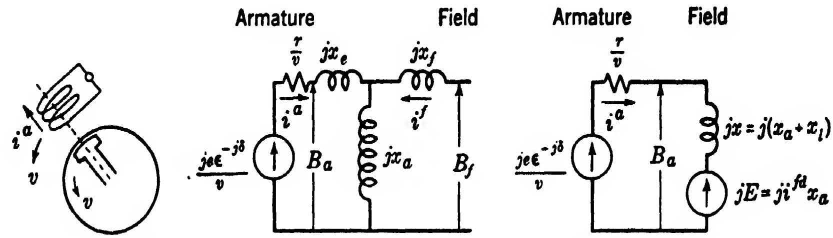
However, $E_Q$ is unknown, whereas the *real* component of $e'$ (or of $E$) is known. Both $e'$ and $E$ may be reproduced on the circuit by adding

and subtracting $x'_d$ (or $x_d$) as shown in Fig. 6.14*a*. The terminating voltage $E_Q$ may now be found by trial and error.

2. Instead of terminating the forward mesh by $E_Q$, it may be terminated by $E_r$, the "voltage behind the *direct*-axis reactance," as shown in Fig. 6.15*a*. All voltages $e'$, $E_Q$, and $E_r$ may be identified on the network.

## POLYPHASE SYNCHRONOUS MACHINES

When the salient pole is absent, $x_{ad} = x_{aq}$ and the impedance in the mutual branch $(x_{aq} - x_{ad})/2$ disappears in all diagrams that represent




(*a*) Rotating frames (*b*) Field mesh included (*c*) Field mesh eliminated

FIG. 6.16. Polyphase synchronous machine. (Synchronous speed operation.)

synchronous speed operation. (During non-synchronous operation the impedance in the mutual branch $[x'_{aq} - x''_{ad}]/2$ does not disappear, since $x''_{ad}$ is not equal to $x'_{aq}$ even when the salient pole is absent.) Since the field voltage $jE$ in the mutual branch can be shifted back from the mutual branch into its original **f** and **b** meshes (Fig. 6.3*a*), *the forward meshes become independent of the backward meshes.*

Since by Eq. 6.1 the phenomena in the **b** meshes are the conjugates of those in the **f** meshes, *the backward meshes may be ignored.* The resultant forward equivalent circuits are shown in Fig. 6.16 for a synchronous machine with smooth airgap running at synchronous speed and excited on both field and armature. The circuit of Fig. 6.16*c* is identical with that of Fig. 6.12*c*, if in the latter figure $x_q = x_d$.

Often the equivalent circuits of several interconnected rotating machines containing salient-pole synchronous machines may be established by first ignoring the saliency and using the circuit of Fig. 6.16*c*. Afterward, the saliency is reintroduced in the form of Fig. 6.12*c*.

# 7 COMMUTATOR MACHINES

## TROUBLESOME EFFECTS IN COMMUTATOR MACHINES

The analytical study of commutator machines is complicated by effects that rarely or never cause trouble in the study of other types of machines. These difficulties may be summarized by the statement that the "constants" of a commutator machine do not remain constant as the speed or the load varies. In particular:

1. Saturation plays an important part. That is, the value of $X_{md}$ and $X_{mq}$ have to be adjusted as the currents vary.
2. Commutation and the effect of coils short-circuited by the brushes influence the machine performance.
3. The non-sinusoidal distribution of windings changes the assumed relations between **B** and $\phi$.

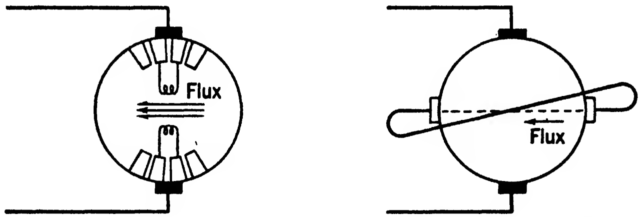
These considerations complicate the work of the designer also, and he must take care of them somehow in any rigorous method of analysis, analytical or graphical.

## COILS SHORT-CIRCUITED BY THE BRUSHES

When a brush covers two commutator bars the coils connected to the two bars become temporarily short-circuited by the brush. When only one set of brushes (per pair of poles) exists on the commutator, the flux due to the two short-circuited coils lies at right angles in space to the flux produced by the other armature coils (Fig. 7.1*a*). This additional flux could just as well have been produced by another (Fig. 7.1*b*) set of short-circuited brushes at right angles to the first set.

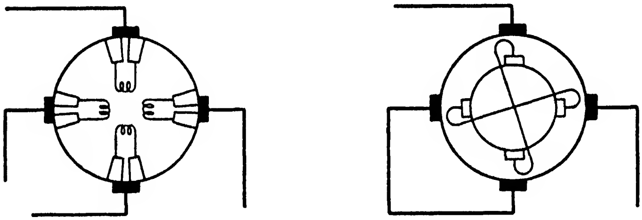
Hence, when only *one set of brushes* exists on the commutator, representing a single rotor winding (not a layer), the short-circuited coils may be replaced by an additional short-circuited winding at right angles in space to the actual winding. This additional winding, however, has a different resistance and may have a different leakage reactance. (Thereby the resultant rotor layer of winding becomes now unbalanced.)

With a *polyphase* set of brushes (Fig. 7.2*a*) existing on all polyphase commutator machines, *the short-circuited coils may be replaced by an additional layer of rotor winding* (Fig. 7.2*b*) below the main winding, having a different resistance and leakage reactance from the actual rotor layer.




(*a*) Shorted coils (*b*) Additional set of brushes

FIG. 7.1. Effect of coils shorted by one set of brushes.



(*a*) Shorted coils (*b*) Additional layer of winding

FIG. 7.2. Effect of shorted coils on a polyphase machine.

## NON-SINUSOIDAL FLUX DENSITIES

Hitherto it has been assumed that, if the rotor velocity $v$ is the same as the speed of the revolving flux (say, unity or synchronous speed), then the generated voltage ($vi$) is equal to the induced voltage ($fi$), or $v = f = 1$. According to this assumption the flux densities are sinusoidal in space. In a-c commutator machines the flux densities, in general, are not sinusoidal, and *at synchronous speed the generated voltage is not equal to the induced voltage.* That is, in commutator machines the **B** vector is not at right angles in space to the $\phi$ vector and is not equal to it (Fig. 2.8). In general, in commutator machines **B** *becomes independent of* $\phi$ *with no apparent relationship between them.*

In order to make the equivalent circuits applicable to cases in which **B** is not proportional to $\phi$, it may be assumed that in each layer of rotor winding the generated voltage is $k$ times the induced voltage. This can be taken care of *by replacing each* $v$ *in the equivalent circuit by* $kv$, where $k$

is different for each layer of winding (or even for each individual winding) and it may be different for each speed $v$.

## GENERALIZATIONS OF THE EQUIVALENT CIRCUITS

In attempting to set up equivalent circuits for commutator machines three new complications arise that require the extension of the equivalent-circuit technique, although none of these facts causes any difficulty in the analytical study. These three new problems are as follows:

1. The *angular displacement* of brushes is similar to an angular displacement of a winding (which also causes difficulties in the shaded-pole motor, Chapter 5).
2. Because of the interconnection of windings and brushes, the *ratio of turns* of the various windings can no longer be assumed to be unity. (In the capacitor motor, Chapter 5, the winding-ratio $a$ was shifted out of the airgap and became part of the impressed voltage and current.)
3. The *series connection* of a stator and rotor winding represents a type of constraint that has no exact analogue in elementary stationary circuit theory. The solution of this difficulty will require delving into the basic concepts of "orthogonal" (or "normalized") stationary networks.

The generalizations of the equivalent circuits hitherto given is undertaken with the understanding that the generalized circuits will be called upon to include all the physical pictures and concepts that the previous ones contained. For instance, it will be required that the voltage across the inductances should give the resultant flux lines linking the windings. Also, it will be required that *the torques should be measured in exactly the same manner as before*, namely, by $\mathbf{i}^*\mathbf{B}$.

In other words, *the equivalent circuits of commutator machines will be as true models of the actual machines as the equivalent circuits of synchronous and induction machines are.* Once the true model circuits have been established they may be modified for other purposes, for instance, to suit the capabilities of the a-c board, or to write the analytical equations of the machine.

The greatest part of this chapter will deal with *polyphase* commutator machines, as the new concepts are simpler and less varied in such machines. The chapter will close with applications to some *single-phase* commutator machines.

## ROTATION OF A POLYPHASE WINDING OR BRUSH SET

When a stator winding or a rotor set of brushes is rotated by an angle $\alpha$ (other windings may exist on the same member), a phase shifter is placed into the vertical branch of the shifted mesh just before the

branch turns horizontal (Fig. 7.3). The current through the resistance and leakage reactance of the equivalent circuit is the same as the current in the machine winding, whereas the current in the mutual react-

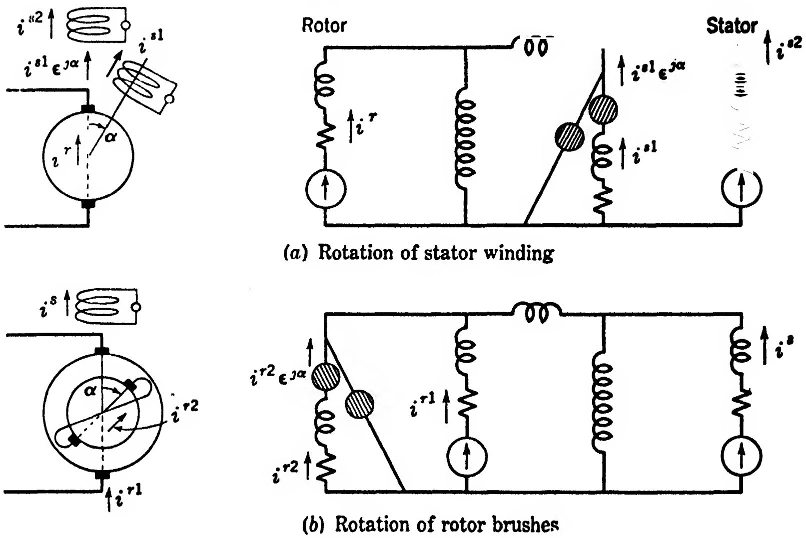



FIG. 7.3. Angular displacement of a reference frame.

ances represents the shifted mmf acting along the direct-axis airgap and not along its original direction.

When two sets of polyphase brushes *connected in series* exist on the same layer of a polyphase machine (Fig. 7.4), then either two separate phase shifters in series are used in that rotor mesh, or only one phase

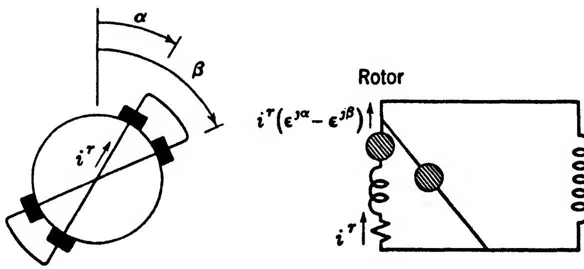



FIG. 7.4. Series-connected brushes.

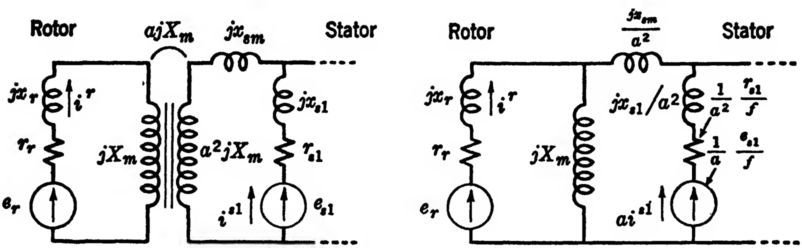
shifter, shifting the currents and voltages by the sum of the brush angles.

Of course, in the final equivalent circuit of many polyphase commutator machines and interconnected machines the phase shifter often

may be shifted out in accordance with the principles shown in Chapter 3. In order not to increase unduly the number of figures in the book, such obvious simplification of the networks will not be shown.

## THE RATIO OF TURNS

When all windings on the stator or rotor structure have an identical number of turns, but a different number from that on the other structure, then the airgap mutual reactance $X_m$ is replaced by an ideal transformer with a turn ratio of $1{:}a$ (Fig. 7.5*a*). Also, all reactances and resistances are multiplied by $a^2$ on the respective structure. (On the a-c board the ideal transformer and the leakage reactances may be combined into an actual transformer.)




(*a*) With ideal transformer (*b*) Transformer eliminated

FIG. 7.5. All stator windings with *a* turns.

If only *one* of the layers of winding has different ratios of turn, then the mesh of that particular winding is electrically isolated from the rest of the network, as shown in Fig. 7.6.

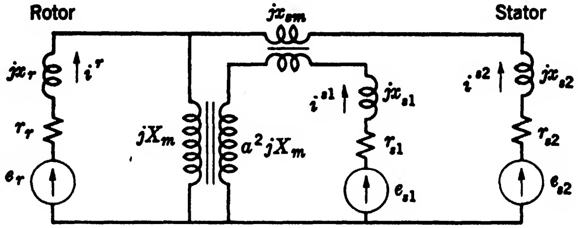


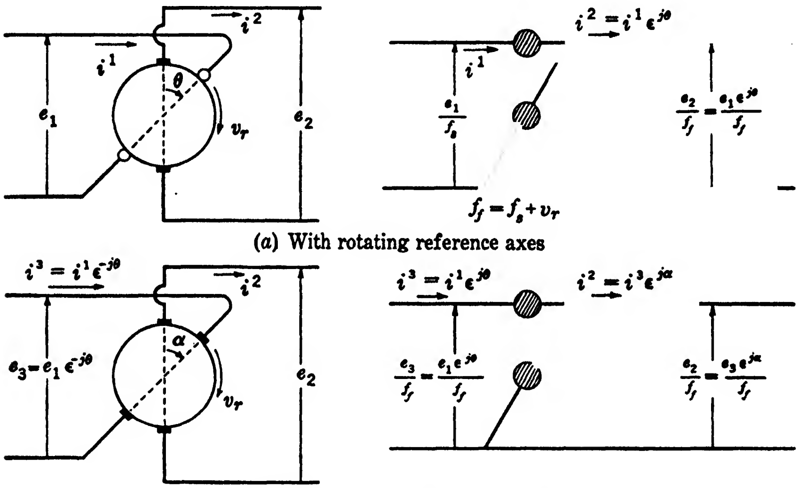

FIG. 7.6. One stator winding with *a* turns.

Just as the phase shifter may often be shifted out of the final network, similarly, the turn ratio may be removed from the resultant network. In that event the ratio of turns *a* disappears from the airgap but reap-

pears in the impedances and in the applied voltages and currents (Fig. 7.5*b*), as explained in Chapter 5.

## FREQUENCY CONVERTER

A simple practical example of an a-c commutator machine is a frequency converter (Fig. 7.7*a*) in which the stator structure is absent. The field impressed on the slip rings rotates with slip speed $f_s$ in the opposite direction of the rotor $v_r$ to induce fundamental frequency $f_f = f_s + v_r$ voltages between the brushes.




(*b*) With stationary reference axes

FIG. 7.7. Frequency converter.

If the slight resistance and reactance drop in the windings is ignored, the equivalent circuit (Fig. 7.7*a*) consists of a phase shifter with a *variable* angle $\theta$, instead of a constant angle $\alpha$. *The absolute frequencies on both sides of the variable phase shifter* (or rather "frequency shifter") *differ by* $v_r$, the speed of the rotor. In particular, on the slip-ring side of the phase shifter the absolute frequency is $f_s$ (slip frequency); hence the impressed voltage $e_1$ is divided by $f_s$, as $e_1/f_s$. On the commutator side the absolute frequency is $f_f = f_s + v_r$; hence the terminal voltage $e_2 = e_1\epsilon^{j\theta}$ is divided by $f_f$, as $e_2/f_f$.

If the slip rings are replaced by stationary brushes (Fig. 7.7*b*) a constant-angular displacement $\alpha$ exists between the two sets of brushes. Its equivalent circuit becomes a phase shifter with a constant angle $\alpha$. The absolute frequencies become identical on both sides of the phase shifter.

## OHMIC-DROP EXCITER

An ohmic-drop exciter is a frequency converter with two sets of series connected brushes on it (per phase) instead of one (Fig. 7.8). In analogy to Fig. 7.4, the structure corresponds to two phase shifters in series, or to one phase shifter that shifts currents and voltages by the sum or dif-

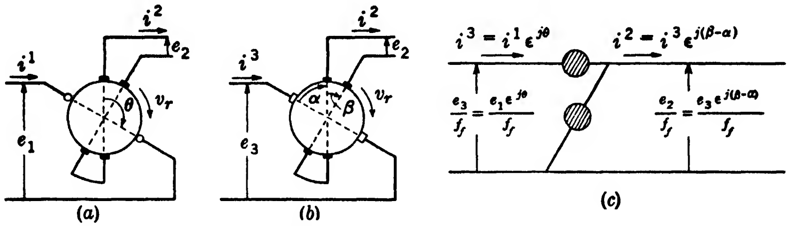



FIG. 7.8. Ohmic-drop exciter.

ference of two angles. The equivalent circuits are identical with those of the frequency converter.

## SHUNT POLYPHASE COMMUTATOR MOTOR

Another simple practical example in which the number of meshes is still the same as the number of layers of windings is the shunt polyphase commutator motor (Fig. 7.9$a$). It differs from the primitive polyphase machine only by having its brushes rotated by an angle $\alpha$ and having $a$

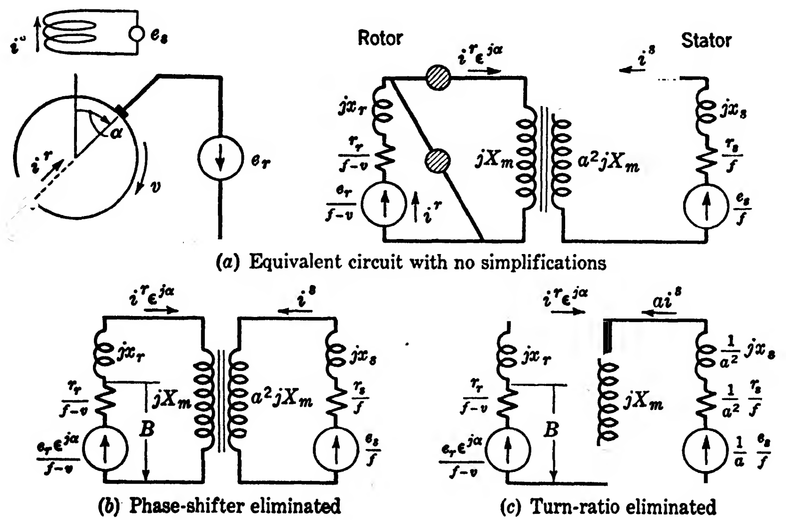



FIG. 7.9. Shunt polyphase commutator motor.

times as many turns on the stator as on the rotor. When the phase shifter is pushed out to the left, the result is as illustrated in Fig. 7.9*b*. When the number of turns is pushed out to the right, the resultant circuit is that of Fig. 7.9*c*. The torque is calculated as usual.

## EFFECT OF SERIES CONNECTION

When a stator and rotor winding are connected in series, two hitherto closed meshes become replaced by one mesh, both in the machine and in its model. This replacement consists in opening a hitherto closed branch. *On the machine* the opening of a branch consists in making a current zero and replacing it by an open-circuit voltage. *On the equivalent circuit* the opening of a branch consists also in making the same current zero and replacing it by an open-circuit voltage. However, as was emphasized in Chapter 2, *the voltages on the equivalent circuits are not equal to the corresponding voltages on the machine* (even though the currents are), since the former voltages correspond to flux densities in the machine and not to the machine voltages.

## THE INTERMEDIARY MACHINE

In order to make the open-circuit voltages identical on both the machine and its model, *an intermediary machine will be introduced* to serve as a bridge between the primitive machine and the actual machine. *This intermediary machine will have as many closed meshes as the primitive machine has* and will be established from the actual machine simply by short-circuiting some of the meshes left open by the series connection.*

It should be recalled that the reason why hitherto it has been so easy to establish the equivalent circuit of any synchronous or induction machine is the single fact that *each machine had exactly as many meshes as it had windings.* The "absolute" frequency of each mesh was the same as that of each winding of the primitive machine. With the introduction of the intermediary machine that same simplicity in the use of absolute frequencies is reestablished for any type of machine.

* It has been shown in *Tensor Analysis of Networks*, by Gabriel Kron, John Wiley & Sons, New York, 1939, p. 428, that to any network that contains fewer meshes than coils it is always possible to add additional meshes or junction pairs in a variety of ways, so that the transformation tensor **C** between the primitive and the intermediary networks is non-singular (square). In stationary-networks analysis, that intermediary step is needed when impressed currents exist or when knowledge of the open-circuit voltage is needed. In establishing stationary models for rotating machines, that intermediary step also becomes an absolute necessity when the machines possess commutators, since a knowledge of the open-circuit voltages again is needed. The same situation arises when several commutator machines are interconnected.

## ISOLATED AND SERIES-CONNECTED WINDINGS

Most commutator machines consist of several isolated windings and of only one or two sets of series-connected windings. Hence *it will be sufficient to set up the equivalent circuits only for such series-connected groups, employing the roundabout reasoning followed for an intermediary machine.* Then the equivalent circuit of the entire machine will be built up of groups of isolated and series-connected windings.

For polyphase machines two such special groups will only be considered:

1. One stator and one rotor winding (both next to the airgap) are connected series-opposing.
2. The same windings are connected series-aiding.

The second case follows automatically from the first; hence only the series-opposing group will have to be considered in great detail.

## STEPS IN THE CONSTRUCTION OF EQUIVALENT CIRCUITS FOR COMMUTATOR MACHINES

The following steps will be made in establishing the equivalent circuits of commutator machines. These steps are valid whether the aim is to establish the equivalent circuit of a series-connected group, of an entire machine, or of several interconnected commutator machines.

1. The equivalent circuit of the *actual machine at standstill* is established.
2. The equivalent circuit of the *intermediary machine at standstill* is next set up simply by short-circuiting some of the junctions of the actual machine.
3. The equivalent circuit of the *intermediary machine under rotation* is established. This step will be analogous to the analysis of the primitive machine in Chapter 3 and will consist in dividing resistances and voltages by the previous absolute frequencies.
4. The equivalent circuit of the *actual machine under rotation* is found next by open-circuiting the shorted junctions.

It will be found that *the open-circuit voltages appear also as impressed voltages elsewhere in the equivalent circuit,* although they do not appear so in the actual machines.

## THE SERIES-OPPOSING GROUP

Let one stator and one rotor winding be connected in series in such a manner that their mutual fluxes oppose each other. The windings have different numbers of turns. The primitive polyphase system at standstill of the two such windings is shown in Fig. 7.10*a*.

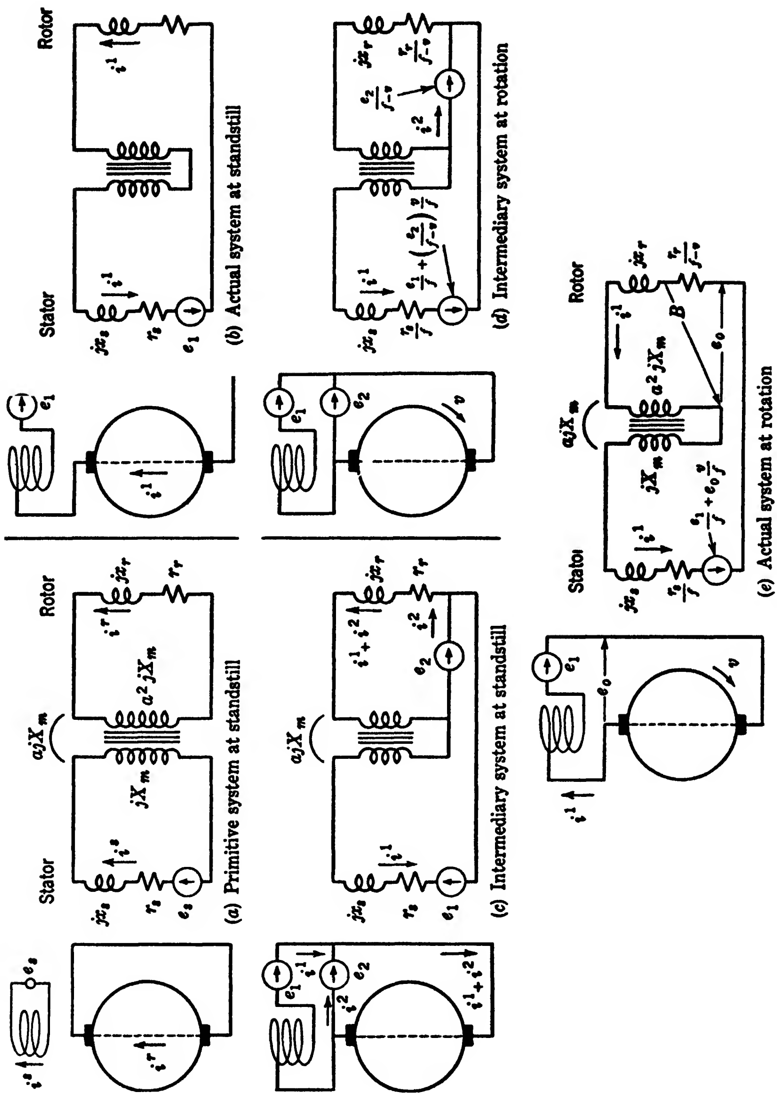



FIG. 7.10. Series-opposing connection of a polyphase machine.

1. The series connection in the equivalent circuit is accomplished in exactly the same manner as in the machine itself (Fig. 7.10*b*). However, *the spatial position of the various horizontal and vertical impedances in the equivalent circuit is left undisturbed and only the impedanceless interconnections are manipulated.* The impressed voltage $e_1$ is assumed to be known. It may be assigned *arbitrarily* either to the stator or to the rotor vertical branch.

It is basic in the philosophy underlying this book that *any manipulation on the equivalent circuit must imitate the manipulations actually performed on the machine.* The series connection of two windings of the machine leaves the configuration of the magnetic field inside the machine undisturbed. Hence, the spatial arrangement of the inductive coils of the equivalent circuit (being a model of the magnetic field) also must be left undisturbed. The manipulation in both cases consists of the series connection of impedanceless wires.

2. The intermediary machine at standstill (Fig. 7.10*c*) is established by a short-circuiting branch in which $i_2$ is assumed to flow and which has an (unknown) applied voltage $e_2$. It should be especially noted that the impressed voltage of the rotor mesh is $e_2$, whereas that of the stator mesh is $e_1 - e_2$. In other words, whatever voltage is impressed around the rotor mesh, the same voltage appears also impressed on the stator.

3. Since the intermediary machine is in effect identical with the primitive machine, the effect of rotation is introduced in exactly the same manner, as shown in Chapter 3. In particular:

(*a*) Divide the rotor mesh quantities $r_r$ and $e_2$ by $f - v$, giving $r_r/(f - v)$ and $e_2/(f - v)$.

(*b*) Divide the stator mesh quantities $r_s$ and $e_1 - e_2$ by $f$, giving $r_s/f$ and $(e_1 - e_2)/f$.

However, $e_1 - e_2$ does not appear as such either in the machine or in the circuit. Both $e_1$ and $e_2$ appear in an isolated manner.

Since $e_2$ reappears divided by $e_2/(f - v)$, the question arises: How should $e_1$ be replaced, as $e_1$ does not represent the *total* impressed voltage in the stator mesh, but only part of it. Summing up voltage drops around the stator mesh and letting $x$ be the unknown form of $e_1$,

$$x - \frac{e_2}{f - v} = \frac{e_1 - e_2}{f}$$

From this equation,

$$x = \frac{e_1}{f} + \frac{e_2}{f - v}\frac{v}{f}$$

*This is the voltage by which $e_1$ is replaced during rotation.* It should be noted that, summing up voltage drops around the stator mesh, the total stator impressed voltage is $(e_1 - e_2)/f$, as it should be for a primitive (or intermediary) machine.

4. If the short circuit is opened up, $i_2$ becomes zero, and *the impressed voltage $e_2/(f - v)$ becomes an unknown open-circuit voltage* (or rather its flux equivalent) $e_0 = e_2/(f - v)$. Although in the actual machine the value of the voltage across the rotor winding $e_2$ is of no interest, in the

equivalent circuit a *knowledge of the voltage $e_0$ across the rotor is needed, being part of the stator impressed voltage.*

Whereas the open-circuit voltage $e_0$ in the actual machine is $e_2$, in the equivalent circuit the open-circuit voltage $e_0$ is the "flux equivalent" of $e_2$; namely, $e_0 = e_2/(f - v)$. This voltage is amplified by $v/f$.

## CALCULATION OF THE OPEN-CIRCUIT VOLTAGE

On the *a-c network analyzer* an amplifier changing $e_0$ to $e_0v/f$ automatically adjusts the stator impressed voltage to its true value.

In employing the equivalent circuit for *writing the equations of performance* of the machine, it is necessary to write first as many equations as there are layers of windings. That is, *the open-circuit voltage $e_0$ (or $e_2$) also is assumed to be an unknown* and the rotor mesh equation must be written as an extra equation for the extra unknown. *If the open-circuit voltage $e_0$ is eliminated afterward,* the resultant equations are the equations of performance that would be found by any correct method of analysis, making the same assumptions.

Of course, it is possible to rearrange the final equations (containing no open-circuit voltages $e_0$) to establish equivalent circuits for commutator machines, as has been done in the past.* But those circuits lack many of the physical attributes that the present equivalent circuits possess. For instance, the torque cannot be visualized on them. Because of their artificiality, conventional equivalent circuits are not suitable to use for interconnecting commutator machines with each other or with other types of machines. The circuits here given have no such limitations.

## THE SERIES-AIDING GROUP

When the fluxes of the stator and rotor windings help each other, the two meshes of the equivalent circuit are interconnected as shown in

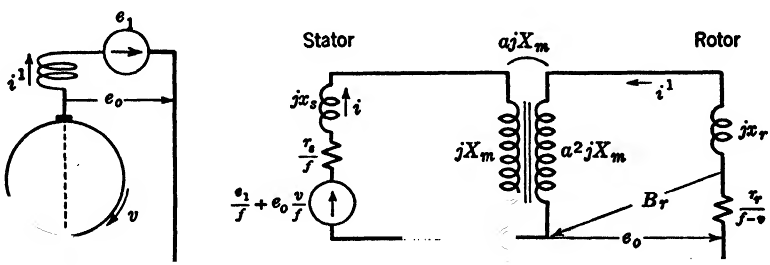



FIG. 7.11. Series-aiding connection.

* W. B. Coulthard, "A Generalized Equivalent Circuit in the Theory of Polyphase Commutator Motors," *Transactions of the AIEE,* Vol. 60, pp. 423–31, 1941.

Fig. 7.11. The steps are *identical with those for the series-opposing group;* so are the final equivalent circuits and impressed voltages.

## THE SCHERBIUS MACHINE

The Scherbius machine contains a series-opposing group and also one or more additional stator windings rotated by 90° from the axis of the series-opposing group (Fig. 7.12*a*). The previously derived series-

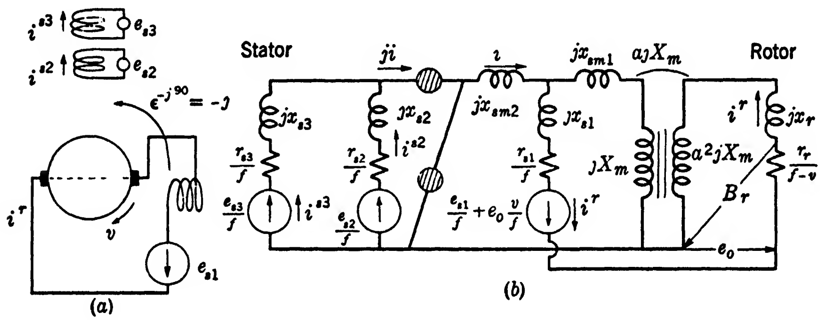



FIG. 7.12. Scherbius machine.

opposing group and two standard stator meshes are combined in Fig. 7.12*b*. The torque is measured as usual $T = i^{r*}Br$.

*The equivalent circuit splits into two independent circuits,* if two simplifying assumptions are made:

1. The mutual reactance $x_{sm1}$ between two stator windings at right angles is zero (Fig. 7.13).

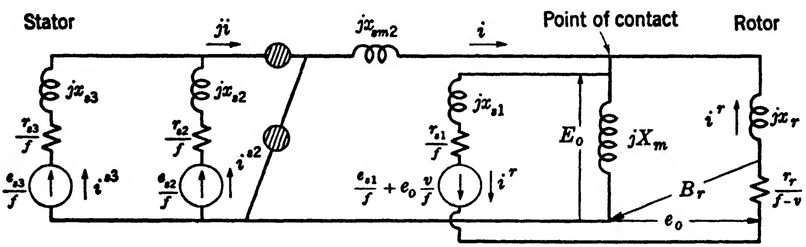



FIG. 7.13. Scherbius machine with complete compensation.

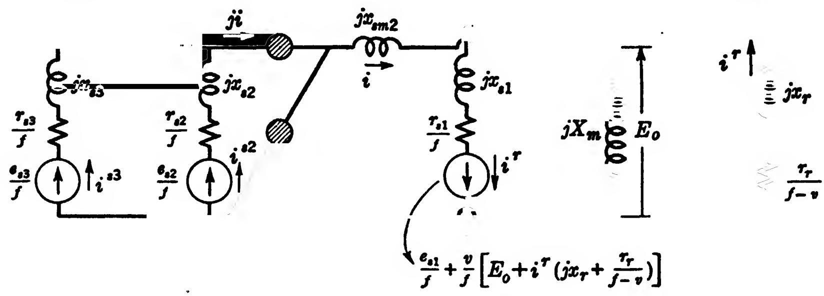
2. The ratio of turns of the two windings in the series-opposing group is unity, and so perfect compensation exists and all airgap flux is produced only by the two shifted windings.

The open-circuit voltage $e_0$ now appears between two electrically independent networks. They are joined together by an impedanceless horizontal branch. This contact can not be broken, as the difference

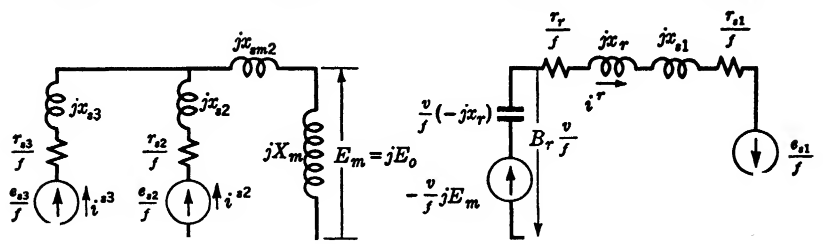
of potential $e_0$ depends on both circuits' having the same absolute potentials at the point of contact.

## TWO ISOLATED EQUIVALENT CIRCUITS OF THE SCHERBIUS MACHINE

The contact point also may be removed by replacing $e_0$ by another open-circuit voltage, $E_0$, appearing across a single impedance $jX_m$ only,




(a) Removal of the contact point




(b) Two independent circuits

FIG. 7.14. Scherbius machine (simplified network).

as shown in Figs. 7.13 and 7.14a. In terms of $E_0$, the open-circuit voltage $e_0$ becomes $e_0 = E_0 + i^r[jx_r + r_r/(f - v)]$.

The impressed voltage expression may be simplified if the last two terms are added to the rotor circuit as negative resistances, leaving only $E_0$ as an impressed voltage. The new rotor resistances are

$$\frac{r_r}{f-v} - \frac{v}{f}\left(\frac{r_r}{f-v}\right) = \frac{r_r}{f}$$

The new rotor leakage inductances are

$$jx_r - \frac{v}{f}jx_r = jx_r\left(1 - \frac{v}{f}\right)$$

If the phase shifter is shifted out *to the right*, then $E_m$, the new voltage measured across $jX_m$, represents $E_m = jE_0$. The open-circuit voltage $E_0$ (to be impressed in the second network) becomes $-jE_m$ in terms of the measurable $E_m$.

The final two isolated meshes are shown in Fig. 7.14*b*.

*Each circuit represents one axis of the machine,* and no amplifier or phase shifter is needed on the a-c board for the study.

## PHYSICAL BARENESS OF THE CIRCUITS

Because of the artificial character of the final circuits, compared with Fig. 7.12 or 7.13, several of the physical attributes have been lost. For instance, the voltage $B_r$ in Fig. 7.13, representing the rotor flux density before the removal of the point of contact, is defined in terms of $E_0$ as

$$-B_r = E_0 + i^r jx_r$$

In terms of $E_m$ it becomes

$$-B_r = -jE_m + i^r jx_r$$

$$-B_r \frac{v}{f} = \frac{v}{f}(-jE_m + i^r jx_r)$$

Hence in the new artificial mesh the voltage across the impressed voltage and the capacitor is $B_r v/f$ and not $B_r$. The torque is still

$$\text{Torque} = i^{r*}B_r$$

where $B_r$ is a measured difference of potential divided by $v/f$.

## SERIES POLYPHASE COMMUTATOR MOTOR

A series commutator motor is shown in Fig. 7.15. Its equivalent circuit is identical with the series-aiding group of Fig. 7.11 with a phase

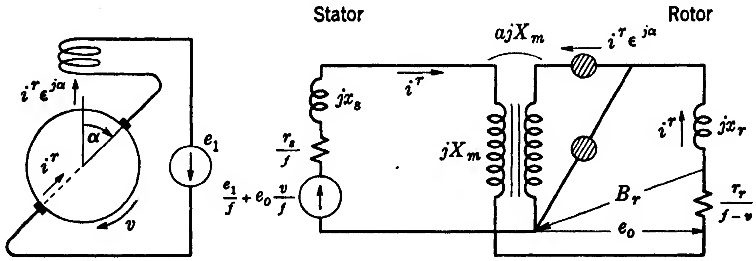



FIG. 7.15. Series polyphase commutator motor.

shifter added across the rotor vertical branch and applied voltage, to represent the shifting of brushes as shown in Fig. 7.15. The torque is

measured as usual by $T = i^{r*}B_r$ across the airgap reactance of the rotor. The phase shifter is canceled in finding $i^*B$, since $\epsilon^{-j\alpha}\epsilon^{j\alpha} = 1$.

## THE SCHRAGE MOTOR

The Schrage motor (Fig. 7.16) is similar in structure to the series polyphase commutator of Fig. 7.15 with the following differences:

1. It has two sets of brushes with two variable angles $\alpha$ and $\beta$, as shown in detail in Fig. 7.4.

2. The voltage is impressed on an additional rotor winding.

The equivalent circuit of Fig. 7.16 incorporates both these features.

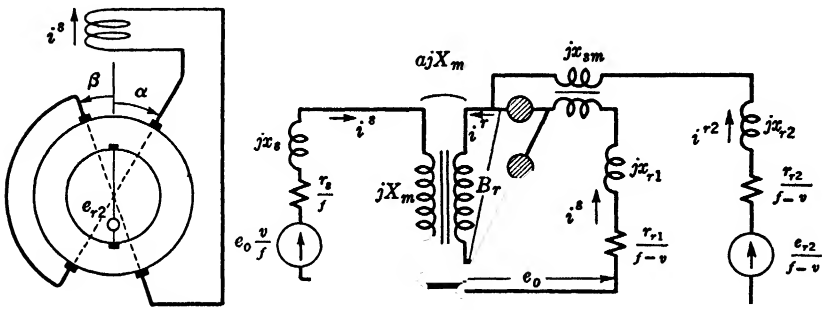



FIG. 7.16. Schrage motor.

## SINGLE-PHASE COMMUTATOR MACHINES

The first two decades of the twentieth century saw the development of a large number of single-phase commutator machines (such as the Déry, Torda, Fynn-Weichsel, etc., motors), which, however, never took root in the commercial field. At present only a few simple types are used in the *power* field, and so a more detailed theory of their equivalent circuits is not justified at this stage. The amplidyne, the rototrol, and, in general, the various metadynes invented by Pestarini * are coming into use in connection with *control* systems, but their theory is outside the scope of this book on *power* machinery.

To show that the basic theory developed for polyphase commutator machines is also valid for single-phase machines, one group of single-phase connections will be treated now, namely, the use of one set of brushes on the rotor, instead of two.

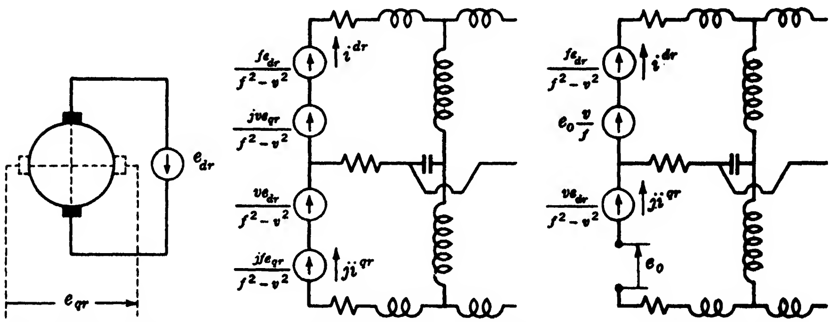
The possibilities for other special groups are great. For instance, examining only series connections, two direct-axis windings—one on the stator, the other on the rotor—may be connected in aiding or oppos-

* J. M. Pestarini, "Théorie du functionnement statique de la metadyne," *Revue Générale de L'Électricité*, Vol. 27, p. 227, March 1930.

ing series; so can two quadrature-axis windings. Both the two windings may lie either on the stator or on the rotor. One of the two windings may be along the direct axis; the other, along the quadrature axis. These series connections may also occur in two or more combinations.

## THE REMOVAL OF ONE SET OF BRUSHES

With both sets of rotor brushes present, the rotor equivalent circuit, by use of the cross-field theory, is shown in Fig. 7.17*b*. Removing the quadrature-axis brush is equivalent to making $i^{qr} = 0$ or to opening




(*a*) One set of brushes (*b*) Both sets of brushes present (*c*) Making $i^{qr} = 0$

FIG. 7.17. Only direct-axis brush-set exists.

the rotor **q** mesh (Fig. 7.17*c*). The rotor open-circuit voltage $e_{qr}$ is unknown ($e_{dr}$ is known); nevertheless, it is still necessary to impress a function of $e_q$ on both rotor meshes of the equivalent circuit. (Their ratio is $v/f$.)

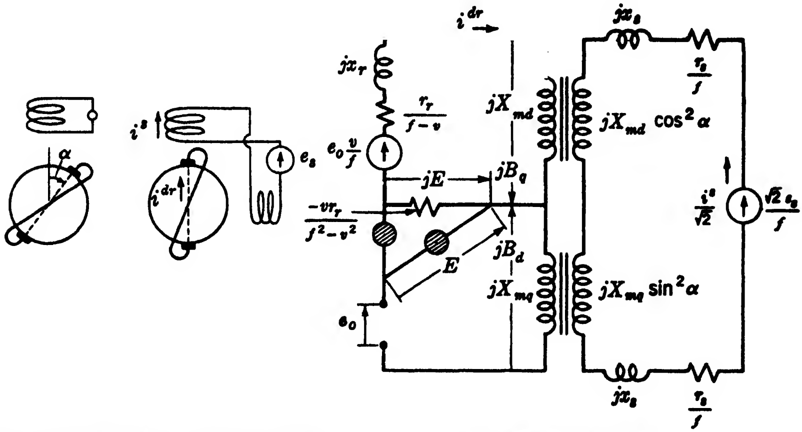
If the open-circuit voltage $jfe_q/(f^2 - v^2)$—a measurable quantity in the **q** mesh of the equivalent circuit—is denoted by $e_0$, then the function of $e_q$, to be impressed on the **d** mesh, becomes $e_0v/f$. Hence Fig. 7.17*c* is the desired equivalent circuit containing an amplifier that measures the unknown $e_0$ in the $\mathbf{q}_r$ mesh and impresses it in the $\mathbf{d}_r$ mesh, amplified by $v/f$.

It should be noted that the amplification factor $v/f$ of $e_0$ is the same as that occurring in polyphase machines.

## THE REPULSION MOTOR

A practical example of a machine with one set of brushes is the repulsion motor, Fig. 7.18*a*. Let the rotor-brush axis be considered the **d** axis (Fig. 7.18*b*). Then the stator winding along the **d** axis contains (by accepted repulsion-motor theory) cos $\alpha$ turns and along the **q** axis sin $\alpha$ turns. Since the two stator windings are connected in series,

$i^{qs} = i^{ds} = i^s$, and it is necessary to use a phase shifter in the rotor to avoid the appearance of $ji^{qs}$ in the equivalent circuit of Fig. 7.18c.




(a) Given machine (b) Assumed axes (c) Cross-field network

FIG. 7.18. Repulsion motor.

## SQUIRREL-CAGE REPULSION MOTOR

If a squirrel cage is added to the rotor of a repulsion motor, the squirrel cage is considered an extra layer of rotor windings with two sets

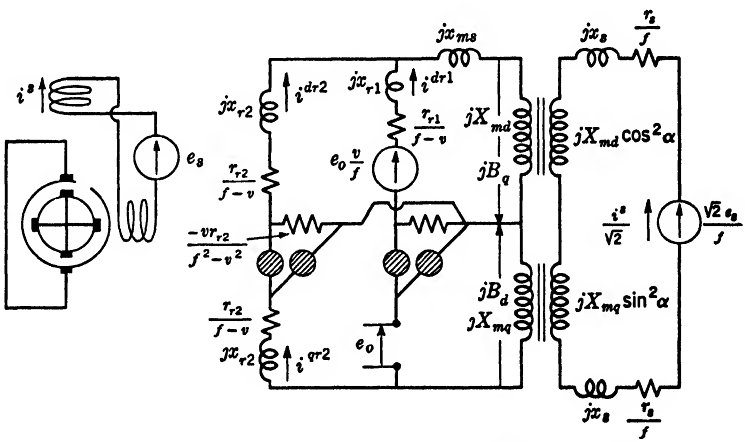



FIG. 7.19. Squirrel-cage repulsion motor (cross-field network).

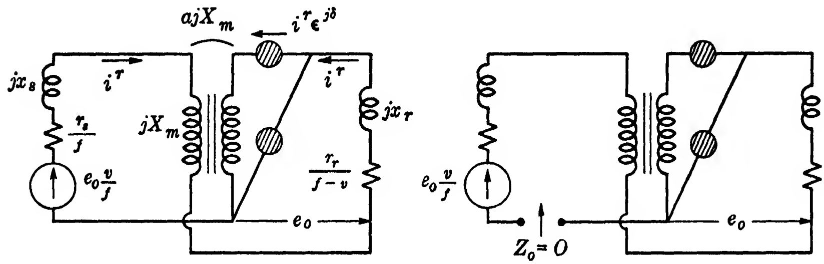
of brushes at right angles in space. The equivalent circuit is shown in Fig. 7.19.

It is interesting and important to note that although the addition of a rotor layer of winding requires only a routine procedure in terms of

equivalent circuits (if the latter are true models and not brute-force substitutes), in the analytical study it complicates the equations considerably.

## SELF-EXCITATION OF COMMUTATOR MACHINES

It is well known that low-frequency currents may exist in commutator machines even when no voltage is impressed on them and no capacitor is connected to them. In such cases the voltages generated by the rotation, $Mvi$, act as capacitive storage of energies. The flux necessary




(a) Oscillatory network (b) Solution for $f$

FIG. 7.20. Self-excitation of the series polyphase commutator motor.

to produce the generation of currents is built up either from remaining magnetization or from some electric impulse.

*If the impressed voltages are removed from all equivalent circuits given,* the *remaining circuit represents the self-excitation network* of that particular machine. An example for the series polyphase commutator motor (Fig. 7.15) is given in Fig. 7.20*a*. *The base frequency $f$ is now an unknown quantity* as it represents the frequency of oscillations that arise. The frequency $f$ may be zero, or it may have several values.

*The value of $f$ may be found by opening up any mesh of the network and calculating the impedance $Z$ of the whole network viewed from the break,* Fig. 7.20*b*, assuming $f$ to be a parameter. The value of $f$ which makes both the real and imaginary components of the impedance zero is the frequency of oscillation.

# 8 STATIONARY NETWORKS

## ARBITRARY CIRCUITS AS TWO-PHASE NETWORKS

In the examples hitherto considered, the stationary networks were connected only to the stator windings of induction motors and consequently they were viewed always from stationary axes. The stationary networks had the additional characteristic of being able to be grouped, without further study, into direct-axis and quadrature-axis impedances. Only the *frequency* of currents flowing through them had to be specially considered for equivalent-circuit representation, by dividing all resistances by $f$ and all capacitors by $f^2$.

The stationary networks to be connected presently to the field structure of a synchronous machine—although still to be viewed from stationary reference frames—cannot be considered as two-phase networks. *The individual amortisseur bars may not be divided outright into two groups of bars, one belonging to the direct axis, the other to the quadrature axis.* New means of viewing stationary networks and defining their meshes will have to be introduced in order to represent an amortisseur winding as a two-phase network, balanced or unbalanced.

## UNEQUAL LOADING OF DAMPER BARS

It is known that the various bars of an amortisseur carry unequal currents; in particular, the trailing edge often carries heavier currents than the leading edge. To avoid unequal heating, the currents flowing in the individual bars must be known.

The purpose of the following study is to replace the amortisseur winding by two sets of coils, one lying along the direct axis, having an overall lumped impedance $x''_d$, the other set of coils lying along the quadrature axis, having an overall impedance $x''_q$ (or $x'_q$). Four types of damper windings will be considered:

1. *Standard* amortisseur winding (complete or incomplete), in which all bars are identical and uniformly spaced around the periphery.

2. *Non-uniform* amortisseur, in which the bars are equally spaced *but each bar has a different impedance.*

3. *Non-uniform and non-symmetrical amortisseur,* in which the bars are unequally spaced and have unequal impedances.

In all cases the bar patterns repeat for each pole.

4. *Solid rotor,* in which the iron structure acts as a damper.

## STANDARD AMORTISSEUR

It has been shown by Linville * that a squirrel-cage winding (Fig. 8.1*a*) may be replaced by two nests of closed meshes (Fig. 8.1*b*), one nest surrounding concentrically the axis of the poles, the other the interpolar space. The vertical lines stand for the squirrel-cage bars; the horizontal lines, for the end rings.

It should be noted that the bars under the right half of the *second* pole have been shifted under the same half of the *first* pole, so that two sets of vertical bars exist there. Since each pole repeats the same configuration of currents and fluxes, the shift is allowable.

The equivalent circuit for each nest of meshes is shown in Fig. 8.1*c*. The circuit has the shape of the squirrel cage it represents, and each branch in the circuit has the same resistance and leakage reactance as the corresponding branch in the squirrel cage. In particular:

1. Each vertical line contains the resistance (divided by the slip $s$) and slot-leakage reactance $jx_b$ of the corresponding bar.

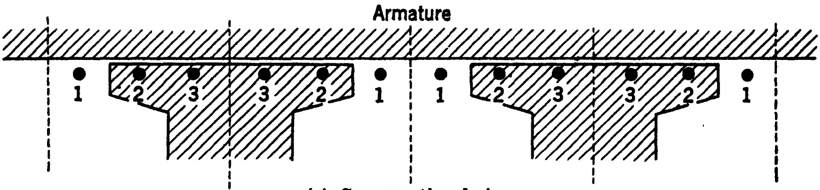
2. The lower horizontal line contains $r/s$ and the leakage reactance $jx_r$ of both corresponding end-ring portions.

3. The upper horizontal line contains the total *fundamental* airgap flux $x_{ad}$ mutual with the armature. This flux is now apportioned between the several meshes, each reactance standing for that portion of the sinusoidal airgap flux that extends within two bars.

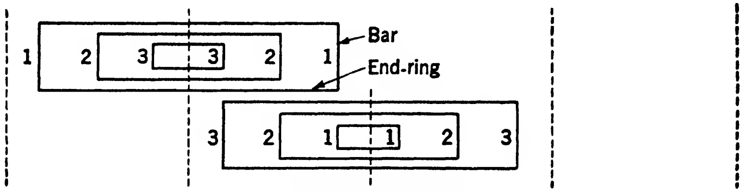
The *field winding* is mutual with the armature mesh through the total $x_{ad}$. However, *the field winding also produces space harmonics* that do not penetrate to the armature but link the various amortisseur meshes. Those mutual reactances between the field mesh and the amortisseur meshes, $jx_h$, must be represented through ideal transformers. (The field mesh has already one *direct* connection with the amortisseur meshes through $x_{ad}$.)

Because of the symmetry of each half pole, the two points $a$ and $b$ are equipotential points. Hence they may be short-circuited and *half of the circuits left out,* as shown in Fig. 8.1*d*. The resultant network represents $x''_d$ of the standard amortisseur (or damper) winding.

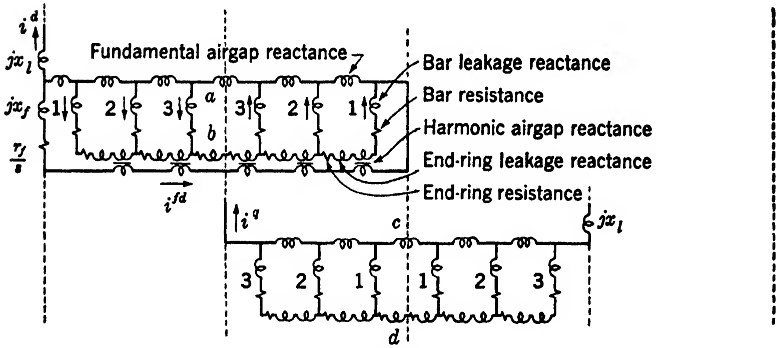
* T. M. Linville, "Starting Performance of Salient-Pole Synchronous Motors," *Transactions of the AIEE*, Vol. 49, p. 535, 1930.



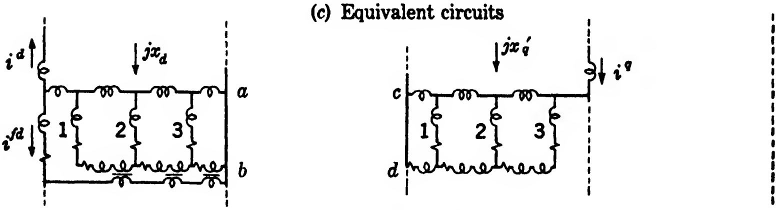

(a) Cross-sectional view




(b) Nested networks

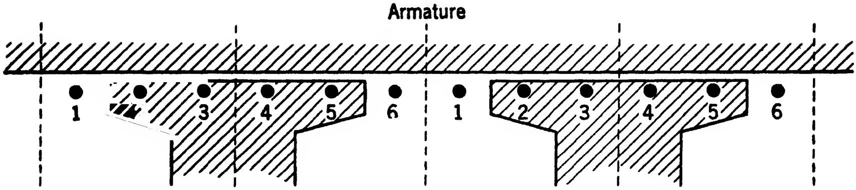



(c) Equivalent circuits

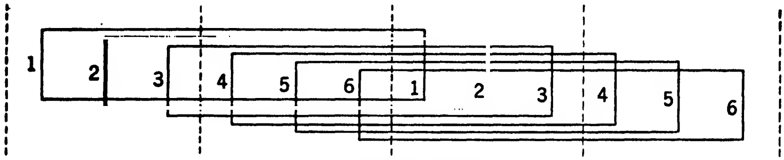


(d) Short-circuit impedances

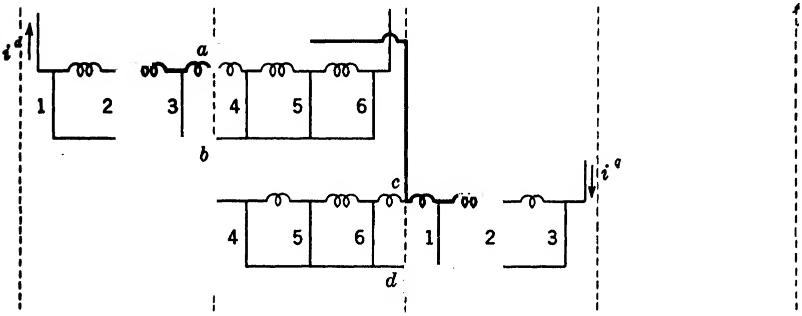
FIG. 8.1. Standard amortisseur.



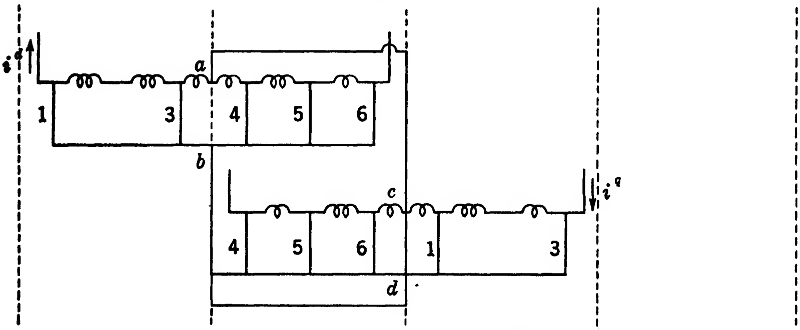

(a) Cross-sectional view




(b) Nested networks




(c) New type of nested mesh




(d) Removing one bar per pole

FIG. 8.2. Non-uniform amortisseur.

Similar development exists for the quadrature axis $x''_q$, except that no field winding exists. Point $c$ and $d$, halfway between the poles, are also equipotential points. In fact, all four equipotential points, $a$, $b$, $c$, and $d$, will be connected to the ground (common branch) in the complete equivalent circuit of the synchronous machine (Fig. 8.3).

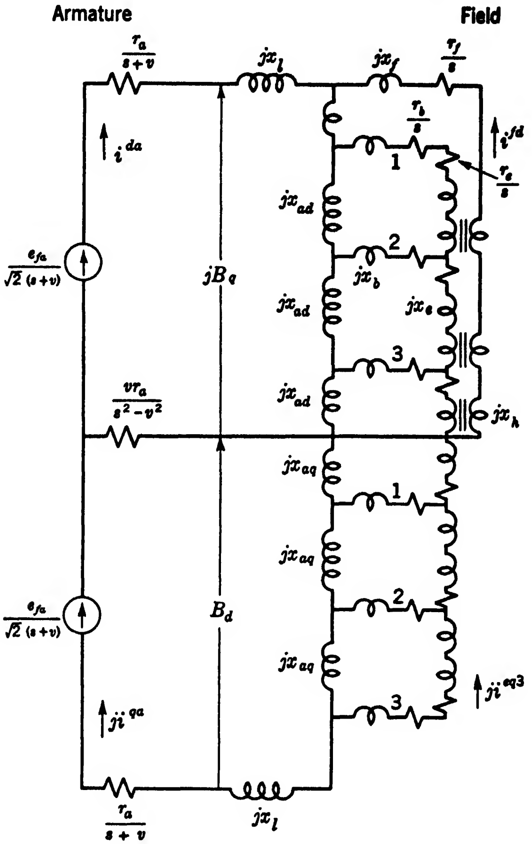



FIG. 8.3. Synchronous machine with standard amortisseur bars (cross-field network, $f = s = 1 - v$).

It is possible to replace the circuits of Fig. 8.1$d$ by two equivalent impedances for each slip $s$ and use them in all circuits of Chapter 6 wherever $x''_d$ and $x'_q$ occur, in analogy to Fig. 6.2.* It is also possible to attach the circuits of Fig. 8.1$d$ to the networks of Chapter 6, as shown in Fig. 8.3, which is identical with Fig. 6.7$b$ except that the amortisseur

* A. W. Rankin, "The Direct and Quadrature Axis Equivalent Circuits of the Synchronous Machine," *Transactions of the AIEE*, Vol. 64, pp. 861–868, 1945.

winding is now represented by a structure with six bars per pole. (It should also be noted that $s + v = f$.)

## A MORE GENERAL DAMPER EQUIVALENT CIRCUIT

The damper equivalent circuit just presented is unable to take care of any asymmetricity along the two halves of one pole, since in the calculation of the reactances *each nested mesh includes two bars that lie in both halves of one pole.*

In order to be able to introduce inequality in the two halves of a pole, it is necessary to replace the squirrel cage by a *new type of nested mesh.* This mesh will consist of two bars 180° apart, say a No. 2 bar under the north pole and a No. 2 bar under the south pole, as shown in Fig. 8.2*b*. Here the bars are renumbered from 1–6 consecutively, instead of from 1–3.

The equivalent circuit of the new type of meshes will again consist of two squirrel-cage type of meshes, one along the **d** axis, the other along the **q** axis, as shown in Fig. 8.2*c*, with the difference that the *center points a-b and c-d of the previously isolated squirrel cages are now interconnected.* Each of the new type of nested meshes may be traced out as shown by heavy lines for mesh No. 2. The reactances on the direct-axis squirrel cage show the direct-axis mutual reactance of this mesh with the armature, whereas the reactances on the quadrature-axis squirrel cage show its quadrature-axis mutual reactances. (The bar-leakage and end-leakage reactances, also the mutual harmonic reactance with the field, are not shown, in order to simplify the drawings.)

It should be noted that, again, the bars under the right half of the second pole have been shifted under the first pole; hence the numbering on the second squirrel cage differs from that on the first cage.

This new equivalent circuit, Fig. 8.2*c*, differs from the previous one, Fig. 8.1*c*, only in the manner in which the four equipotential points *a*, *b*, *c*, and *d* are interconnected. In the previous one *a-b* and *c-d* are shorted; here, however, *a-c* and *b-d* are shorted. Hence, both circuits must give the same answer, and there is no difference, discernible as yet, between them.

## BARS WITH NON-UNIFORM IMPEDANCES

If in the amortisseur winding two bars of a pair of poles (similarly situated under each pole) are removed, then *the corresponding two bars in the equivalent circuit* (forming one nested mesh) *can also be removed as* shown for the No. 2 nested mesh in Fig. 8.2*d*. The remaining nested meshes all can be traced with their correct mutual reactances. (In the analogous network of Fig. 8.1*c* this removal cannot be effected.)

If, instead of removing the set of No. 2 bars, their resistance or leakage reactance (located in the equivalent circuit on the vertical bars themselves) is made to differ from those of the other bars, a similar change is made on both branches of the equivalent circuit. In general, in the network of Fig. 8.2*d*, each set of bars or end rings may have different resistances and leakage reactances.

The unbalancing of the two squirrel cages introduces a difference of potential between points *a* and *b* (or *c* and *d*), lying at the center of a pole. Consequently, a voltage impressed on the direct-axis circuit produces also currents in the quadrature axis. In other words, a nonuniform amortisseur on the field structure is analogous to the stator structure of a shaded-pole induction motor (Figs. 5.19 and 5.20). *In both machines the* **d** *and* **q** *axes of the salient structure are mutually coupled even when the machine is at rest.*

## ASYMMETRICALLY PLACED BARS

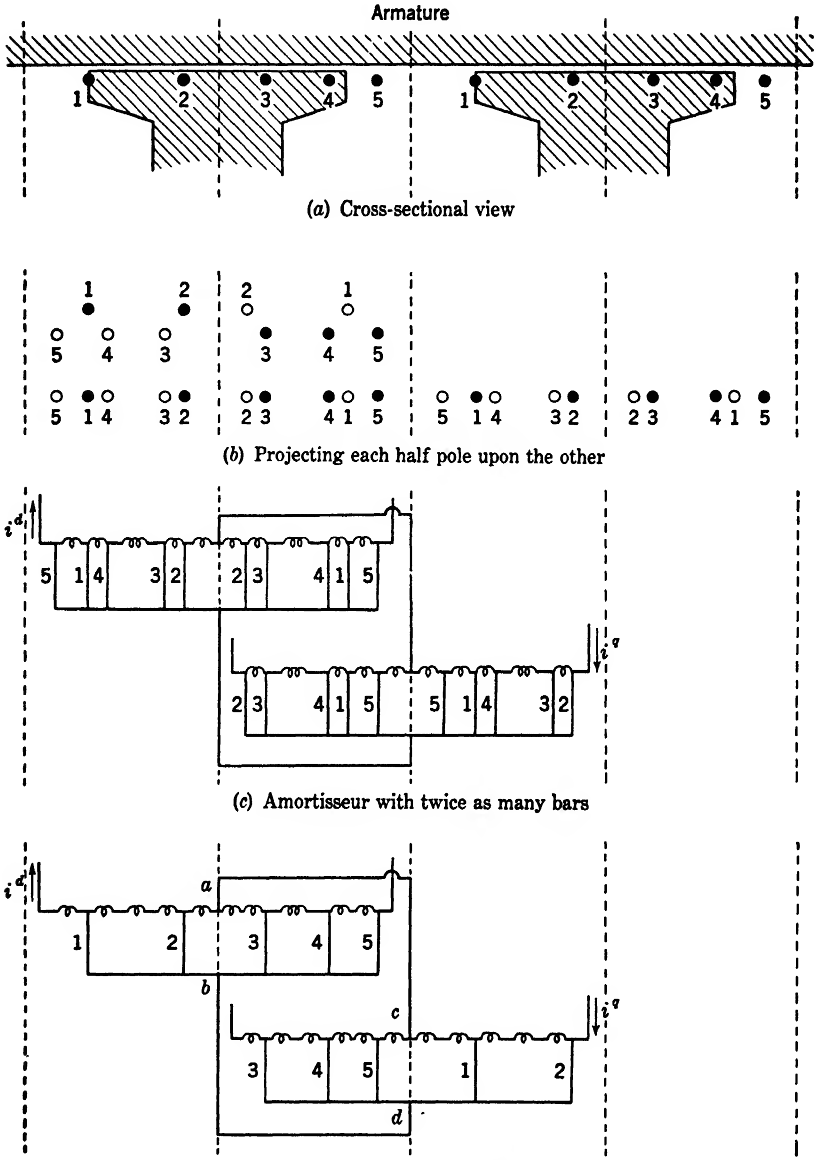
The methods of calculating the reactances of nested meshes all are based upon the assumption that the bars are symmetrically spaced with respect to the center of the pole. When the bars are asymmetrically spaced (Fig. 8.4*a*), the problem arises how to use the already-known method of reactance calculations of Fig. 8.1 for the asymmetrical amortisseur of Fig. 8.4. The method of attack is as follows:

1. *Assume twice the number of bars* that actually exists, by projecting each half pole on the other half (on Fig. 8.4*b* the second set of hypothetical bars is shown by small circles).
2. Since the double amortisseur looks like a standard amortisseur having twice as many bars, all its constants may be established by standard procedures.
3. Establish the standard equivalent circuit of the double damper, Fig. 8.4*c*, using all of the direct and quadrature squirrel-cage networks. The field winding is added as usual.
4. Connect the equipotential points of the two circuits as shown in Fig. 8.4*c*.
5. Remove all bars that belong to the hypothetical half (small circles) (Fig. 8.4*d*).

The resultant circuit gives $x''_d$, $x'_q$, and $x'_{dq}$. The complete equivalent circuit with the armature excited is shown in Fig. 8.5.

## FIELD REPRESENTATION OF A SOLID ROTOR

In Fig. 6.2*c* a solid rotor was represented by four amortisseur windings arranged in concentric circles. A more correct representation of a solid rotor is in the form of a two-dimensional network satisfying the field




(a) Cross-sectional view

(b) Projecting each half pole upon the other

(c) Amortisseur with twice as many bars

(d) Removing excess bars

FIG. 8.4. Non-symmetrical amortisseur.

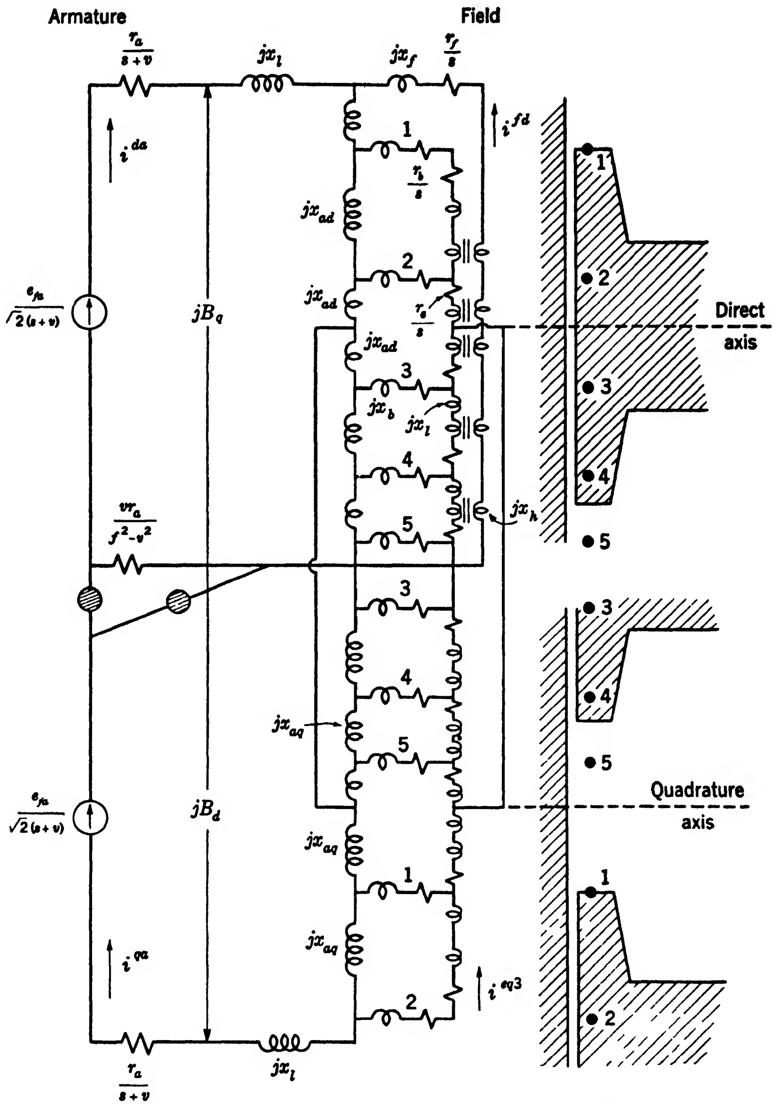



FIG. 8.5. Synchronous machines with non-symmetrical amortisseur bars (cross-field network, $f = s = 1 - v$).

equations of Maxwell for a region containing magnetic flux densities $B$, field intensities $E$, and current densities $I$.* When one portion of a machine is considered to be a circuit, and another portion to be a field, the equivalent circuit of each part must be so defined that the portions can be interconnected.

It will be shown that the analogy between the equivalent circuit representation of an entire rotating machine and the two-dimensional circuit representation of a small portion of the machine is complete, and therefore the two types of circuits may be interconnected into one circuit.

## THE EQUIVALENT CIRCUIT AS A ONE-DIMENSIONAL FIELD

Let several concentric layers of windings be assumed on the rotor, as shown schematically (with one slot per pole) on Fig. 8.6*a*. (Compare with Fig. 2.4*b*.) Assuming no impressed voltages and no end-leakage fluxes, the resultant rotor equivalent circuit along the **d** axis is shown in

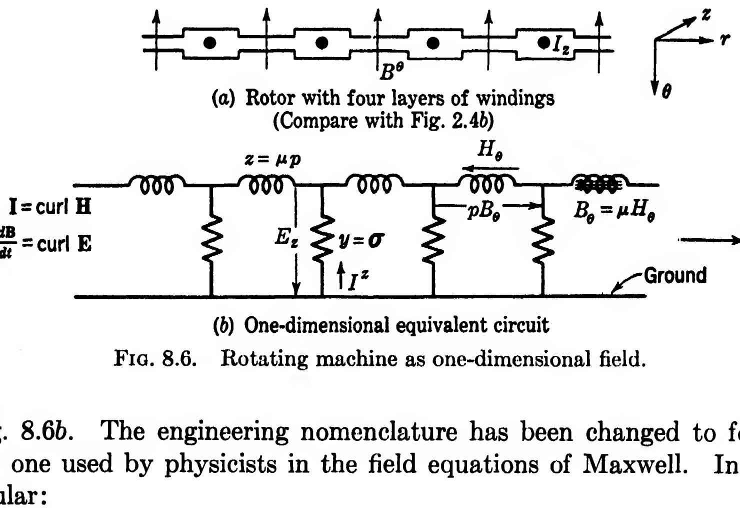



FIG. 8.6. Rotating machine as one-dimensional field.

Fig. 8.6*b*. The engineering nomenclature has been changed to follow the one used by physicists in the field equations of Maxwell. In particular:

1. The inductance $L$ becomes permeability $\mu$.
2. The resistance $r$ becomes conductance $\sigma = 1/r$.
3. The actual currents $i$ in the windings are represented by the current $I^z$ in the *vertical* branches.
4. The resultant currents in the horizontal inductors represent the resultant mmf's $H_\theta$ producing the fluxes in the $\theta$ direction.

* Gabriel Kron, "Equivalent Circuit of the Field Equations of Maxwell—I," *Proceedings of the Institute of Radio Engineers*, Vol. 32, pp. 289–299, May 1944.

5. The flux $B^\theta = \mu H_\theta$ produced by each inductor is equal to the flux passing perpendicularly between two layers of windings.

6. The voltage induced in each inductor is $pB^\theta$.

7. The difference of potential $E_z$ induced in each winding produces the winding currents $I_z = \sigma E_z$.

## ONE-DIMENSIONAL FIELD EQUATIONS OF MAXWELL

*The direct-axis equivalent circuit of Fig. 8.6b may be looked upon as the one-dimensional equivalent circuit of a one-dimensional magnetic field* (Fig. 8.6*a*), in which the direction of the flux line ($B^\theta$) is $\theta$, the mmf ($H_\theta$) distribution varies in the radial direction, and the electric-field intensity $E_z$ is axial (along $z$). The field equations of Maxwell are as follows:

1. The magnetic field intensity (mmf $= H_\theta$) distribution in the radial direction is discontinuous because of the existence of axial currents,

$$-\frac{\partial H_\theta}{\partial r} = I^z \qquad \Bigg| \qquad \operatorname{curl} \mathbf{H} = \mathbf{I}$$

2. The electric-field intensity $E_z$ distribution in the radial direction is also discontinuous because of the voltages induced by the time variation of flux lines,

$$\frac{\partial E_z}{\partial r} = -\frac{\partial B^\theta}{\partial t} \qquad \Bigg| \qquad \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

The quadrature-axis equivalent circuit satisfies a similar set of equations. Hence the primitive-machine equivalent circuit considers a rotating machine to be two coupled one-dimensional magnetic fields at right angles in space.

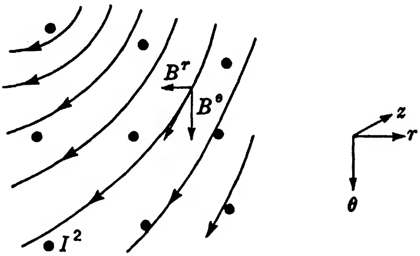
## TWO-DIMENSIONAL FIELD EQUATIONS OF MAXWELL IN POLAR COORDINATES

The equivalent circuit of Fig. 8.6 may be generalized to two dimensions for a solid rotor by replacing the concentric layers of windings by concentric conductors, so that variation in mmf ($H$) exists not only in the radial direction ($r$) but also in the tangential direction ($\theta$) as shown in Fig. 8.7*a*. The currents $I^z$ represent the eddy currents in the solid rotor.

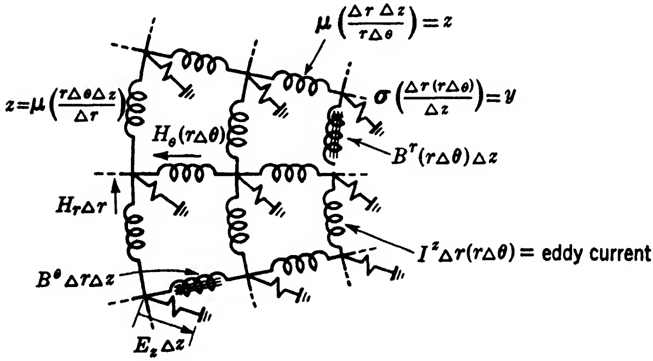
Because the permeability $\mu$ and conductivity $\sigma$ of infinitesimal cubes having the iron section of one mesh in Fig. 8.7*b* vary with the area of penetration and also with the direction in which the cube is penetrated

by the flux or current, the inductances and resistances in the equivalent circuit of Fig. 8.7*b* vary with the radius $r$, as shown. Also it should be noted that:

1. The mmf's $H$ and voltages $E$ represent *line integrals* along an infinitesimal distance.
2. The fluxes $B$ and currents $I$ represent *surface* integrals along an infinitesimal area.




(*a*) Two-dimensional flux distribution




(*b*) Equivalent circuit in polar coordinates

FIG. 8.7. Two-dimensional equivalent circuit of a solid rotor.

## THE INTERCONNECTION OF FIELDS WITH CIRCUITS

Let the cross-section of a solid rotor under one pole have a wedge shape, as shown in Fig. 8.8 by means of a two-dimensional field network. If no eddy currents are assumed to flow (no amortisseur exists), the resistances to ground are infinite and the inductive network represents the distribution of the airgap flux in the rotor. The reactance of the whole network viewed from the two edges of the outer arc must be the total direct-axis airgap reactance $x_{ad}$.

When eddy currents flow and the network is connected to the ground through the resistance, the impedance measured from the two edges of

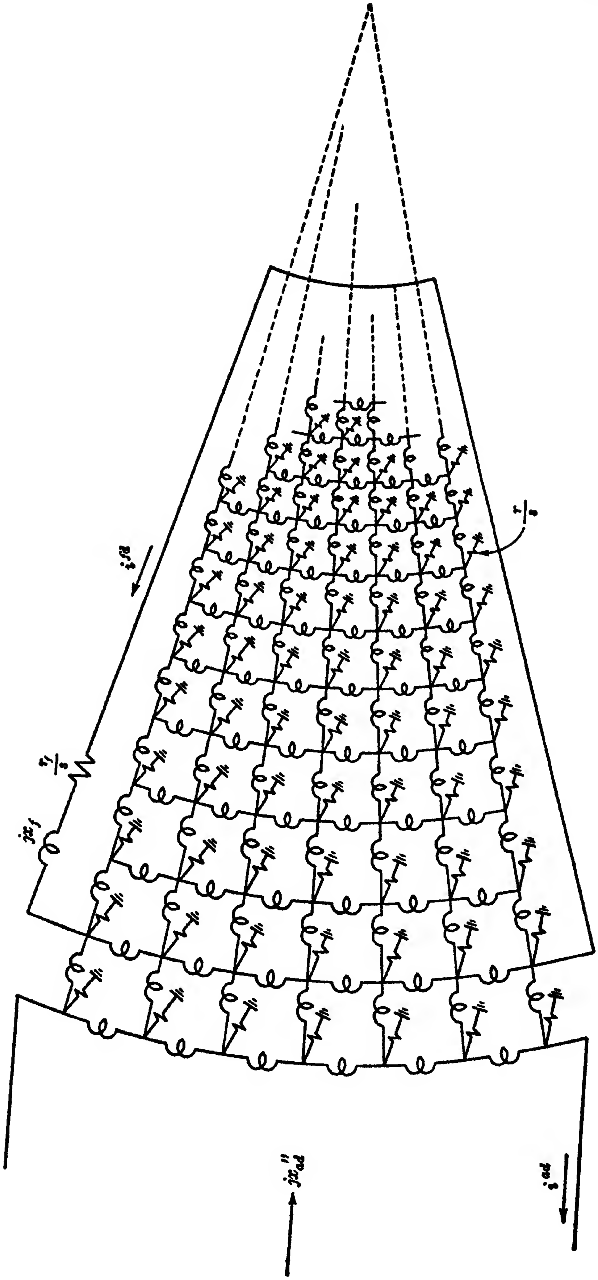


FIG. 8.8. Direct-axis equivalent circuit of a portion of a solid rotor.

the outer arc is the short-circuit impedance $x''_{ad}$. Since Eq. 3.2, for the armature mesh, has been divided by the absolute frequency $f + v$, the *equations of all field meshes also must be divided by their absolute frequency*, which is $f$ for each mesh. Hence *all resistances in this wedge-shaped circuit must be divided by* $f$.

The mesh of the field winding may be directly coupled to the armature mesh or, to simulate the distance effect of the field winding from the armature surface, may lie at a distance from it, as shown. Depending on the accuracy of representation, slot openings also may be inserted in the wedge-shaped network, and their manner of coupling to the field winding may also be refined.

The quadrature-axis airgap reactance $x_{aq}$ of the armature mesh is also coupled to a similar wedge-shaped network. But here the coupled field mesh is absent. The total impedance measured from the armature mesh gives $x'_{aq}$.

## STATIONARY NETWORKS ALONG ROTATING AXES *

When the customary equations of stationary networks are written, it is usually forgotten, but nevertheless implied, that the *reference axes are also stationary*. However, when the reference axes rotate, the equations of the stationary networks assume quite different forms.

There are two important cases for which the customary equations of stationary networks must be extended, because of the presence of rotating axes:

1. Stationary networks connected to the slip rings of induction motor rotors. Since these rotor reference frames assumed are usually stationary in space (**d** and **q**), nevertheless the reference frames rotate backward with respect to the slip-ring brushes; hence they rotate backward also with respect to the stationary networks connected to the slip-ring brushes.

2. Stationary networks connected to the stationary armature of synchronous machines. Since the armature reference frames assumed usually rotate with the rotor, they also rotate with respect to the stationary networks and transmission lines connected to the armature.

## THE RELATIVITY OF MOTION BETWEEN REFERENCE FRAMES AND MATERIAL BODIES

Since the resistance and leakage inductance of the *rotor* of an induction motor may also be looked upon as balanced two-phase impedances lying *outside* the rotor, the manner of appearance of these impedances in the

* Gabriel Kron, "Stationary Networks and Transmission Lines Along Uniformly Rotating Reference Frames," *Transactions of the AIEE*, pp. 690–696, Vol. 68, 1949.

primitive machines of Figs. 3.1 and 3.6 suggests the method of representing any other resistance and inductance connected to the slip rings. Figure 8.9 shows thereby the representation of two-phase stationary resistors and inductors along uniformly rotating reference frames, assuming the sign convention of the primitive induction machine.

*Expressed in another way, the equivalent circuits of balanced stationary networks along rotating reference frames are the same as those of rotating machines along stationary reference frames.*

The analogy is valid only for the equation of voltage, though. For instance, a rotating machine viewed from a stationary reference frame exerts a torque, but no torque appears when a stationary network is being viewed from a rotating reference frame.

## STATIONARY CAPACITORS ALONG ROTATING REFERENCE FRAMES

Since rotating machines contain only resistances (representing the *electric* circuits) and inductances (representing the *magnetic* circuits), their primitive networks of Figs. 3.1 and 3.6 give no clue to how to represent capacitors (*electrostatic* circuits) along rotating reference frames.

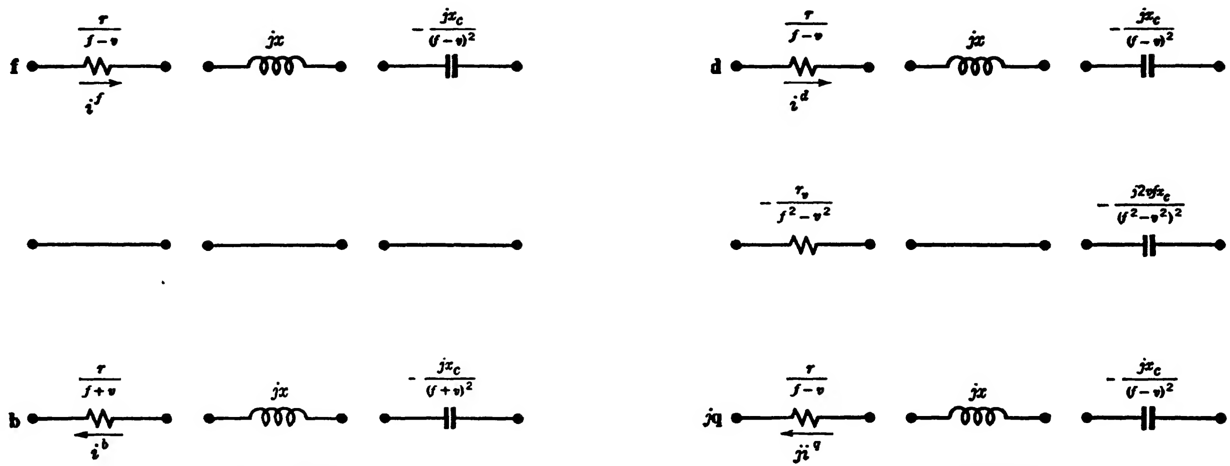
However, the concept of "absolute frequency" comes to the rescue. In the revolving-field networks, the denominator of the rotor resistances gives the absolute frequencies of currents passing through the rotor and thereby through any capacitor connected to the rotor. Hence, in writing the rotor voltage equation for the forward mesh, the capacitor term $C/p$ becomes $-jx_c/(f - v)$ just as $Lp$ becomes $jx(f - v)$. Dividing the voltage equation by the absolute frequency, *the capacitors appear divided by the square of the absolute frequencies*, as $-jx_c/(f - v)^2$, whereas the resistances are divided by the absolute frequencies themselves. (The inductance terms lose their absolute frequency factor by the division.) Figure 8.9*a* shows the representation of a two-phase balanced capacitor along a rotating reference frame, expressed along the hypothetical sequence axes.

The step from the revolving-field equivalent circuit to the cross-field equivalent circuit follows Eqs. 3.5 and 3.7. For instance, the term in the mutual branch is the difference between the two terms in Fig. 8.9*a*.

$$\frac{1}{2}\left[\frac{-jx_c}{(f-v)^2} - \frac{-jx_c}{(f+v)^2}\right] = \frac{-2vfjx_c}{(f^2 - v^2)^2}$$

## SELF-EXCITATION OF INDUCTION AND SYNCHRONOUS MACHINES WITH CAPACITOR LOADS

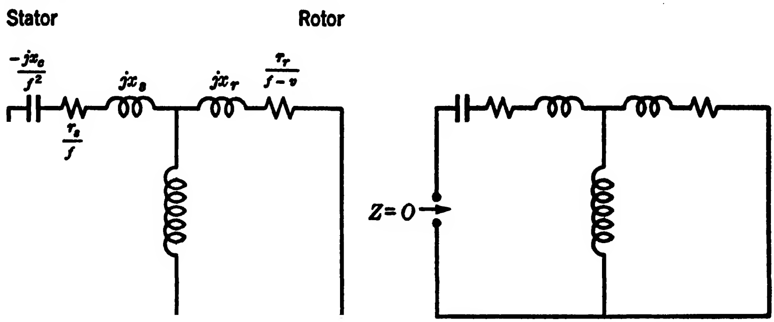
Whereas commutator machines self-excite without the addition of capacitors (Chapter 7), induction and synchronous machines self-excite




(a) Revolving field (b) Cross field

FIG. 8.9. Balanced two-phase impedances along uniformly rotating reference frames (primitive induction-machine sign convention).

only when capacitors are inserted into their circuits. *The self-excitation network for any machine is found simply by removing the impressed voltages,* allowing *the base frequency $f$ to become an unknown quantity.* Self-

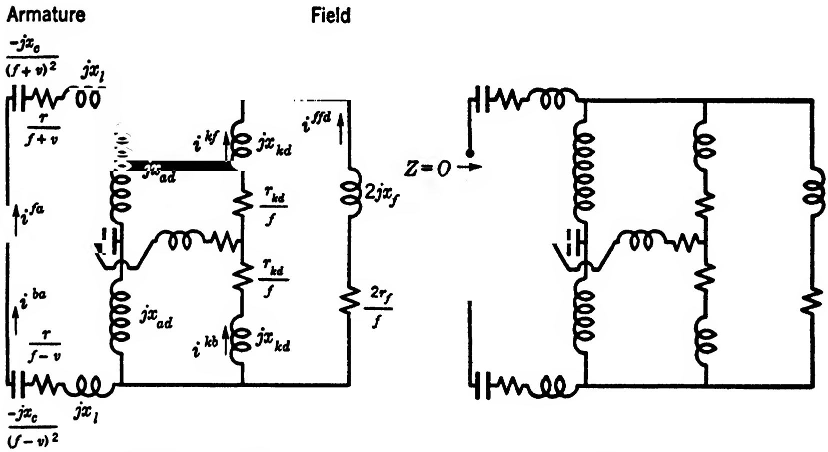



(a) Oscillatory network (b) Solution for $f$

FIG. 8.10. Self-excitation of the polyphase induction motor.

excitation networks of a polyphase induction motor and a synchronous machine are shown in Figs. 8.10$a$ and 8.11$a$.

The method of solution for the frequency of oscillation $f$ is shown in Figs. 8.10$b$ and 8.11$b$, where the impedance $Z_0$ of the circuits viewed from




(a) Oscillatory network (b) Solution for $f$

FIG. 8.11. Self-excitation of the salient-pole synchronous machine.

the break need be calculated and its real and imaginary components equated to zero. *The value of $f$ for $Z_0 = 0$ is the frequency of oscillation.* It may have several values, or it may be zero.

# 9 INTERCONNECTED MACHINES

## PRINCIPLES OF INTERCONNECTION

The equivalent circuits of rotating machines and stationary networks are interconnected in arbitrary groupings in exactly the same manner as the actual machines and networks. Systems in shunt or in series are duplicated by equivalent circuits in shunt or in series. The basic principles to observe are the following:

1. *The meshes to be interconnected all must be expressed along exactly the same reference axes*, physical or hypothetical.

2. The absolute frequencies, $n$, along the meshes to be interconnected must be the same.

3. The base frequencies, $f$, in every machine should be identical. (Thereby in *writing* the transient equations of the equivalent circuit all $f$'s may be replaced by $\dot{p}/j$.)

The interconnection of the equivalent circuits may be accomplished, however, along a reference axis different from that of the interconnection of the actual machine, as long as the two or more reference axes to be interconnected are identical. For instance, the stators of actual *synchronous machines* are interconnected along stationary **d** and **q** axes, and their equivalent circuits may be interconnected along the rotating **d** and **q** axes, or along the hypothetical **f** and **b** axes. Similarly, the rotors of *induction machines* are joined together along the rotating axes (slip rings), and their equivalent circuits may be combined along stationary axes.

Since the slip rings are rigidly attached to the rotor of an induction motor and rotate with it, the phrases "slip-ring axes," "axes rotating with the rotor," and "axes rigidly attached to the rotor" express identical types of axes.

## THE INTRODUCTION OF ROTATING REFERENCE FRAMES (SLIP RINGS) IN POLYPHASE MACHINES

Whenever rotating machines with slip rings are connected together through their slip rings, the interconnection of their equivalent circuits is greatly simplified if the latter are also expressed along the reference frames attached to the slip rings. *The transformation from the conventional to the slip-ring reference frames will be performed with the aid of*

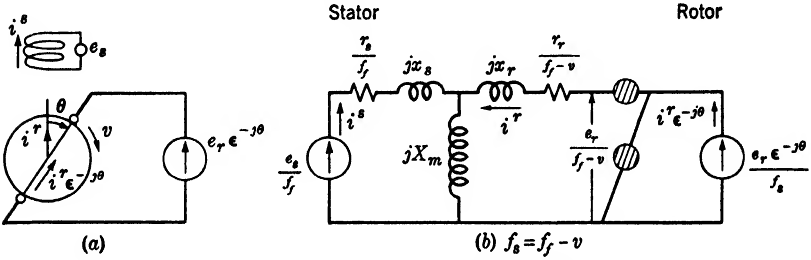



FIG. 9.1. Polyphase induction motor with rotating reference frame on the rotor.

*"variable" phase shifters, whose angles θ are not constant* but functions of time. However, in the interconnected system all these variable phase shifters will either disappear or be replaced by phase shifters with constant angles, after they have been moved about and combined.

Assuming at first only polyphase machines, the previous equivalent circuit of a polyphase induction motor, Fig. 5.4, is generalized to Fig.

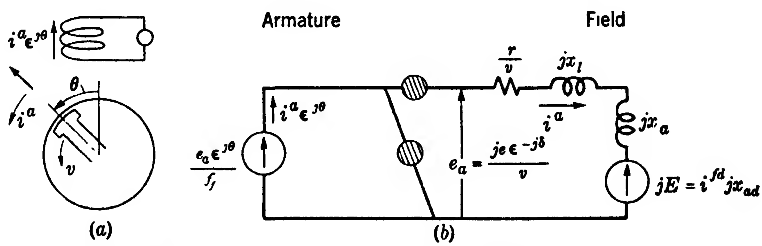



FIG. 9.2. Polyphase synchronous machine with rotating reference frame on the armature.

9.1 by adding a variable phase shifter to the *rotor* terminals. The old current $i^r$ represents the rotor currents along the stationary **d** axis, and the new current $i^r\epsilon^{-j\theta}$ represents the slip-ring current actually leaving the wound rotor terminals. The base frequency (the frequency along the assumed reference frame) on one side of the phase shifter is fundamental frequency $f_f$; on the other side, it is slip-frequency $f_s$.

The standard equivalent circuit of the synchronous machine, Fig. 6.12, is generalized in Fig. 9.2. The old armature current $i^{fa}$ is along the axis rotating with the field, and the new current $i^{fa}\epsilon^{j\theta}$ flows actually in the stationary armature windings. The base frequencies are zero and fundamental, respectively.

## THE TRANSFORMATION FROM STATIONARY TO ROTATING REFERENCE FRAMES

As these polyphase phase shifters are moved about, not only the currents and voltages are shifted by an angle $\epsilon^{-j\theta}$, Fig. 9.3*b*, but also all stationary network and rotating machine resistances passed over change from their stationary reference frame values to their rotating frame values (or vice versa). *The transformation of reference frames consists in adding or subtracting v in the denominator of the resistance or impressed voltage passed over* (as shown in Fig. 9.3), depending on the sign of $\theta$,

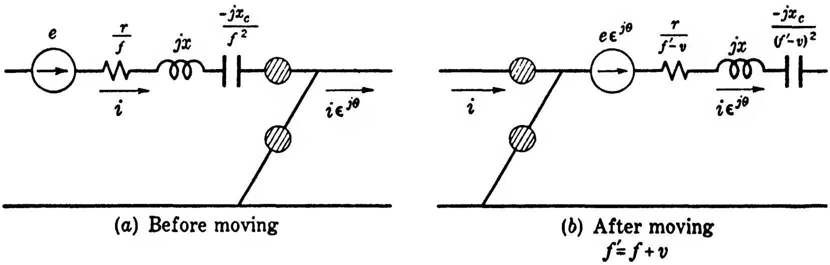



FIG. 9.3. Moving a variable phase shifter.

*and changing simultaneously the value of the base frequency f by v*, so that the absolute frequency remains unchanged.

If the resistances and impressed voltages passed over have already been divided by a function of $v$, the new transformation simply adds an additional $v$, namely, the speed of the reference frame. This $v$ usually differs in value from the existing $v$'s.

It should be recalled from Chapter 3 that the absolute frequency has been defined as the frequency of the currents in the conductors. Such a frequency should be independent of the reference frame assumed (that is, such a frequency is "invariant"). As the variable phase shifter is moved about, the expression for the absolute frequency $n$ varies from a value such as $f_1 - v_1$ to a value $f_2 - v_1 + v_2$. However, the new base frequency $f_2$ also will assume a different form, $f_2 = f_1 - v_2$, so that the final numerical value of the absolute frequency $n$ remains unchanged at the value $n = f_1 - v_1 = f_2 - v_1 + v_2$ and will always give the correct frequency in the conductors.

Summarized, the effect of the introduction of stationary and rotating reference frames in the same machine is to introduce several base frequencies $f$ in the equivalent circuits without changing the numerical values of the absolute frequencies $n$.

## FREQUENCY CONVERSION AND REFERENCE-FRAME ROTATION

The transformation from stationary to rotating reference frames (with the aid of a variable phase shifter) differs from the introduction of an actual frequency converter, Fig. 7.7 (also represented by a variable phase shifter) in the values of the absolute frequencies appearing on both sides of the phase shifter. *In a tranformation of reference frame (a mathematical procedure) no change in the absolute frequency may occur;* hence in Figs. 9.1 and 9.2 on both sides of the phase shifter the same absolute frequency occurs. However, a frequency converter is an actual dynamo machine, and it does change absolute frequencies; hence in its equivalent circuit different absolute frequencies occur on the two sides of the phase shifter.

## TWO POLYPHASE INDUCTION MOTORS (SELSYNS)

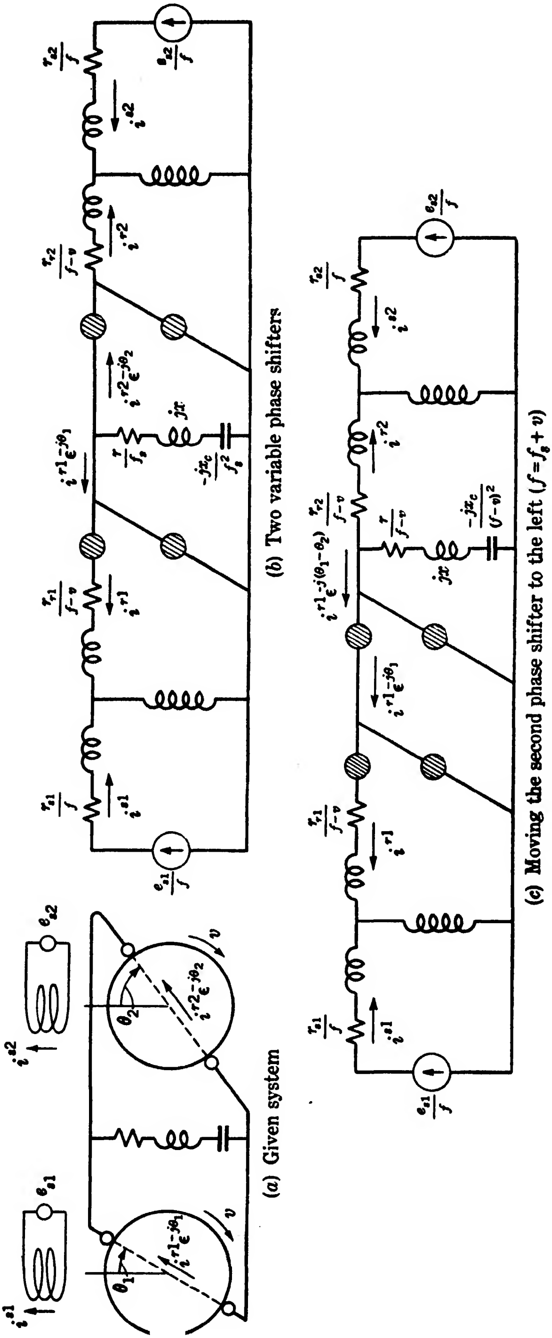
Let two polyphase induction motors be interconnected through their slip rings (Fig. 9.4*a*), with a shunt load across the rotors. When one of the rotors is driven, the other will run at the same speed with a constant angular displacement $\delta = \theta_2 - \theta_1$ between the two rotors ($\theta$ is the instantaneous displacement of each rotor). On the stator of each machine the same base frequency $f$ is impressed.

Figure 9.4*b* gives the equivalent circuits of the resultant system in the form of a primitive system plus two phase shifters with variable angles. The load is still expressed as a *stationary* network along *stationary axes* (or rather as a rotating network that appears stationary with respect to a reference frame rotating with the slip rings). In the stationary load the frequency of currents is slip frequency $f_s$.

Figure 9.4*c* shows the second variable phase shifter transposed next to the first one. Since it passes across the load, *the stationary network must be changed over to its rotating reference frame form by* changing all $f_s$ frequency to $f - v$ frequency, where $f - v = f_s$.

Figure 9.4*d* shows the two variable phase shifters combined into a single phase shifter with a *constant* angle $\delta = \theta_2 - \theta_1$.

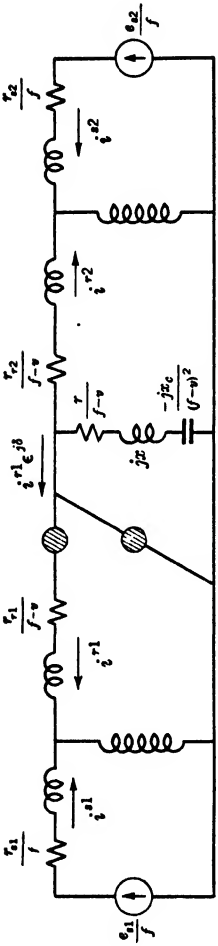
Figure 9.4*e* shifts the constant phase shifter to the left. The impedances now do not vary, but the currents and voltages passed over are shifted by $\epsilon^{j\delta}$. This equivalent circuit expresses the resultant system along stationary rotor reference axes, as shown in Fig. 9.4*f*. The elimi-



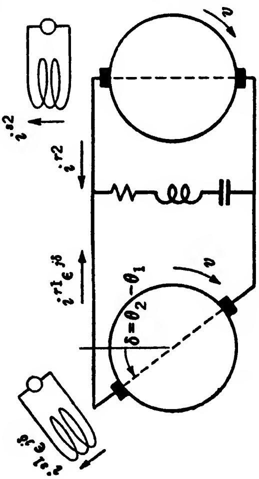

(a) Given system

(b) Two variable phase shifters

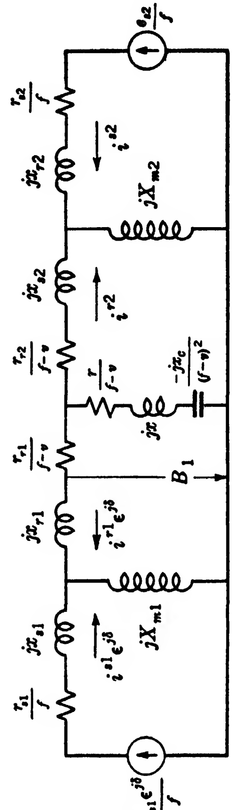
(c) Moving the second phase shifter to the left ($f = f_s + v$)



(d) Replacing two variable by one constant phase shifter



(e) Reference frames of circuit f

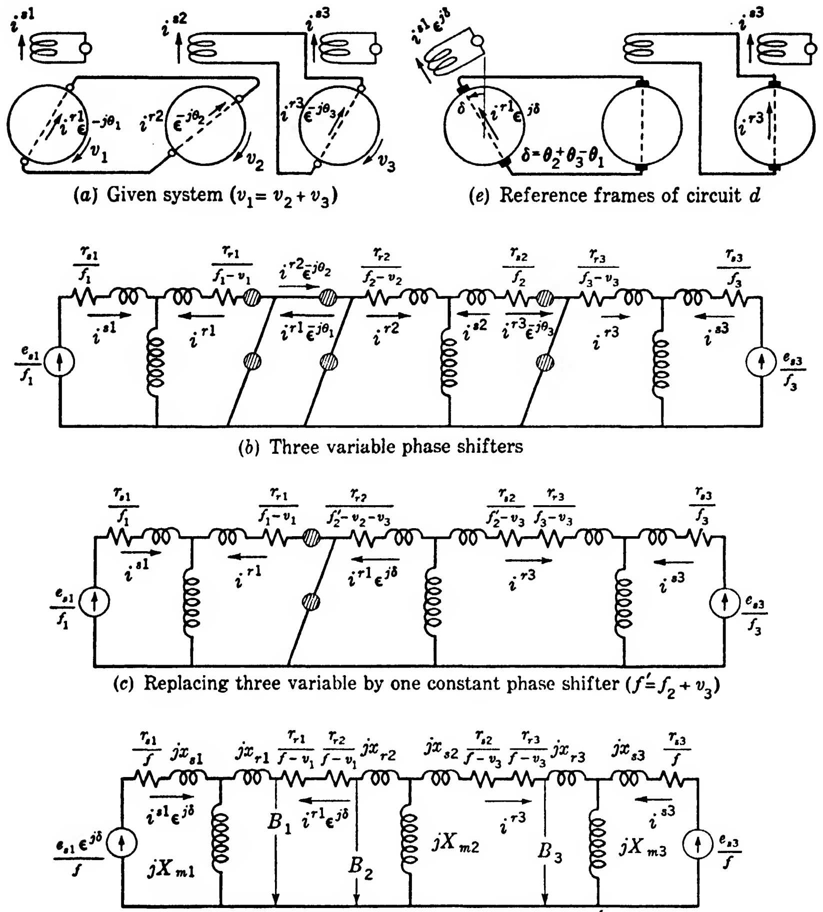


(f) Eliminating the constant phase shifter

FIG. 9.4. Interconnection of two polyphase induction motors (selsyns).

nation of the constant phase shifter is equivalent to shifting the stator reference axis of the first motor by the constant angle $\delta$.

Both the speed $v$ of the rotors and the angle of the lag $\delta$ are variable parameters.



(a) Given system ($v_1 = v_2 + v_3$)

(e) Reference frames of circuit $d$

(b) Three variable phase shifters

(c) Replacing three variable by one constant phase shifter ($f_2' = f_2 + v_3$)

(d) Eliminating the constant phase shifter ($f_1 = f_2' = f_3 = f$)

FIG. 9.5. Differential selsyns.

## THREE POLYPHASE INDUCTION MOTORS (DIFFERENTIAL SELSYNS)

Let three induction motors be connected as in Fig. 9.5*a*. If the second and third rotors are driven at $v_2$ and $v_3$ speed, respectively, the first rotor runs at the *sum* of $v_2$ and $v_3$; that is, $v_1 = v_2 + v_3$. In the three stators the frequencies of currents are assumed to be different, namely, $f_1$, $f_2$, and $f_3$.

The equivalent circuit of the resultant system is shown in Fig. 9.5*b*. The system differs from the previous one (Fig. 9.4*b*) in the important respect that, *as the variable phase shifter* $\theta_3$ *is moved to the left, it passes across an entire motor.* Hence, *all terms in the absolute frequencies in the second motor change* by $v_3$ ($f'_2 = f_2 + v_3$).

The three variable phase shifters combine into one constant phase shifter with angle $\delta = (\theta_2 + \theta_3) - \theta_1$. This can be shifted out of the circuit to the left as shown in Fig. 9.5*d*. This last circuit represents the stationary reference frames of all three machines shown in Fig. 9.5*e*. Hence *all base frequencies* $f_1$, $f_2$, *and* $f_3$ *become unity along these axes.*

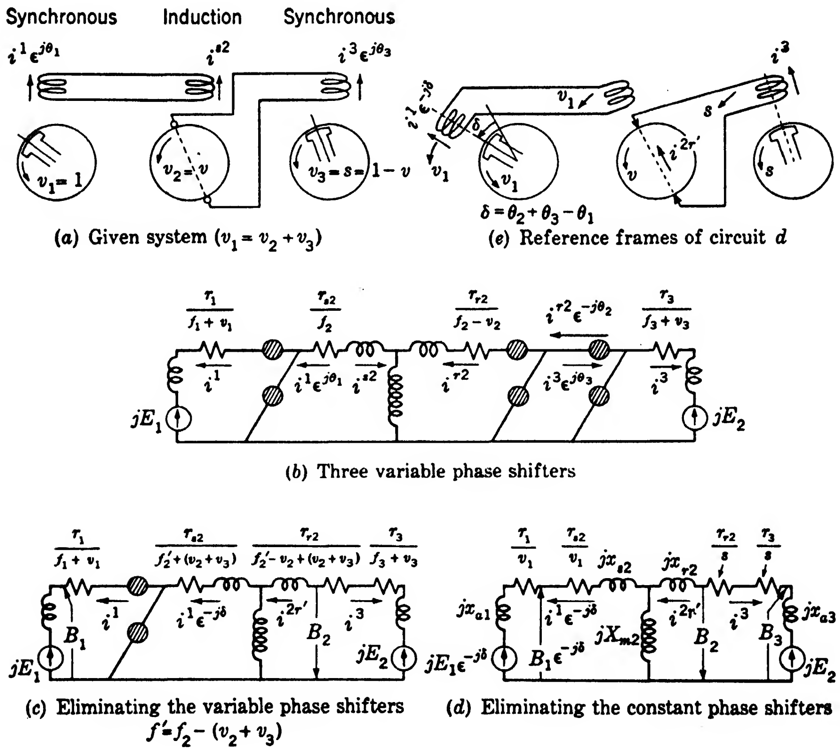
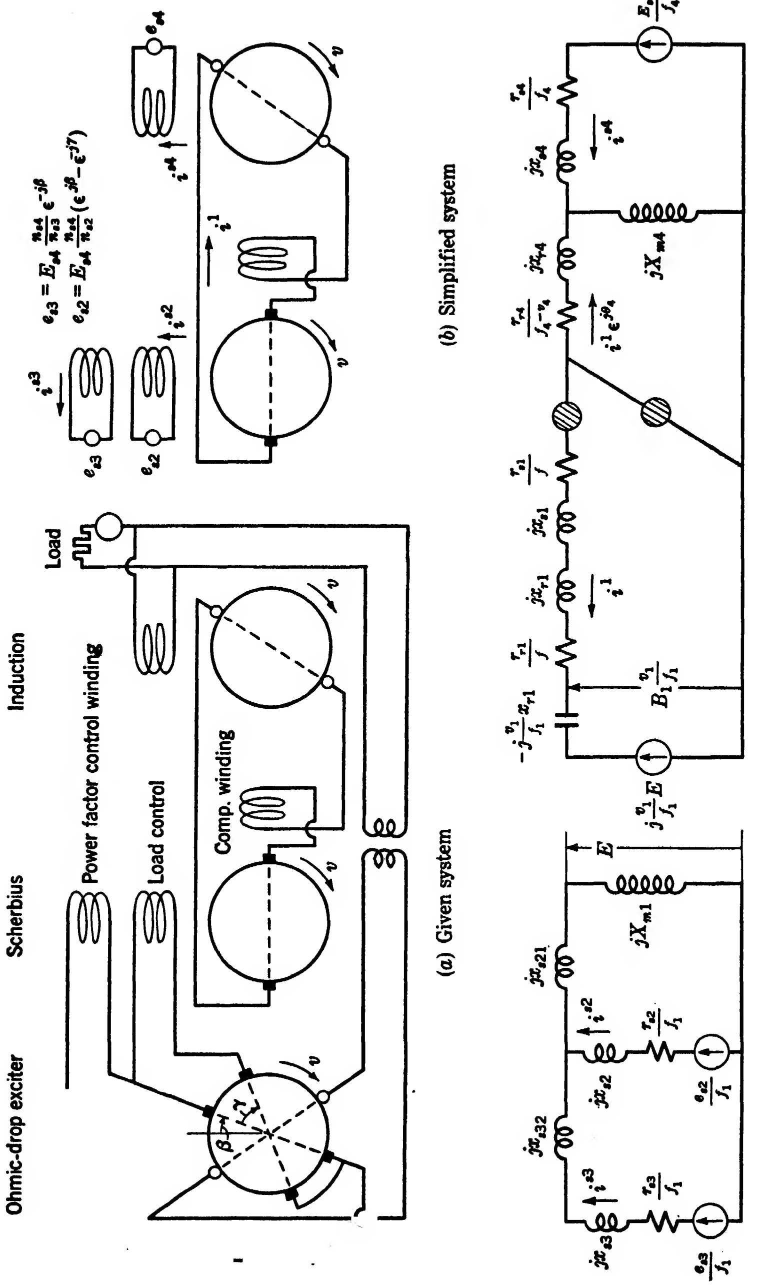



FIG. 9.6. Wind-tunnel fan drive.

## INTERCONNECTION OF POLYPHASE SYNCHRONOUS AND INDUCTION MACHINES

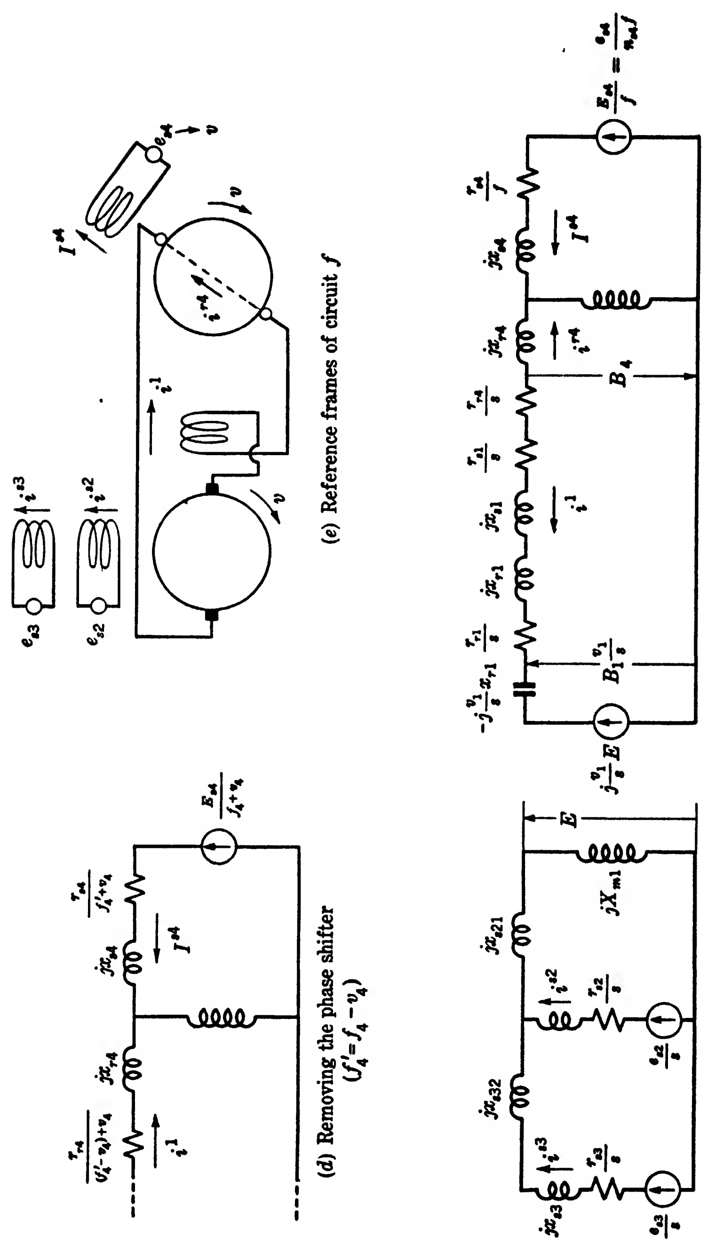
In some variable-speed drives (Fig. 9.6*a*) the stator of an induction motor is supplied from an alternator, and the slip rings of the rotor excite the stator of a synchronous motor. The sum of the speeds of the




(a) Given system

(b) Simplified system

(c) One variable phase shifter



(d) Removing the phase shifter ($f_4' = f_4 - v_4$)

(e) Reference frames of circuit $f$

(f) Replacing $f_1 = f_4' = f - v = s$

FIG. 9.7. Variable-ratio frequency changer.

induction motor $v_2$ and the synchronous motor $v_3$ is that of the alternator $v_1$ = unity; that is, $v_1 = v_2 + v_3$.

The equivalent circuit of the resultant system is shown in Fig. 9.6*b*. As the two variable phase shifters $\theta_2$ and $\theta_3$ are moved to the left, they pass across the induction motor. Hence all absolute frequencies in the induction motor *increase* by $v_2 + v_3$. (Note that the signs of $\theta_2$ and $\theta_3$, in the moving phase shifters, have changed from the previous sign of $\theta_3$ in Fig. 9.5*b*.) By shifting $\epsilon^{j(\theta_2+\theta_3)}$ across the whole induction motor, *its stationary reference frame has been changed into a frame rotating synchronously with the flux in both stator and rotor.*

The three variable phase shifters again combine into one constant phase shifter with angle $\delta = \theta_2 + \theta_3 - \theta_1$. This can be shifted out of the circuit to the left, as shown in Fig. 9.6*d*. Now the reference axes in all three machines rotate with the fluxes (Fig. 9.6*e*), so that *the base frequencies $f_1$, $f_2$, and $f_3$ all become zero.* Both $v$ (or $\delta = 1 - v$) and $\delta$ are variable parameters.

When the synchronous motor has saliency, the resulting equivalent circuit is given in Fig. 9.19 in connection with a numerical example.

## VARIABLE-RATIO FREQUENCY-CHANGER SET

Two power systems with different frequencies may be tied together through synchronous or asynchronous ties. An example of an asynchronous tie is shown in Fig. 9.7*a*, representing a so-called variable-ratio frequency-changer set. It consists of an ohmic-drop exciter, a Scherbius machine (regulating machine), and an induction machine.

The effect of the ohmic-drop exciter is to shift the voltage $e_s$ impressed upon its slip rings by two different angles. The portion shifted by $\beta$ through utilization of only *one* set of brushes is employed for power-factor control, whereas the portion shifted by the *two* series-connected sets of brushes is used as a load control. Hence, by impressing $e_s\epsilon^{-j\beta}$ and $e_s(\epsilon^{-j\beta} - \epsilon^{-j\gamma})$ in place of $e_s$, the presence of the ohmic-drop exciter is taken care of (Fig. 9.7*b*).

The remaining system consists of a Scherbius machine and an induction motor with their rotors connected in series. The equivalent circuit of the systems is shown in Fig. 9.7*c*. If the variable phase shifter is moved to the right, the absolute frequency of the induction motor increases by $v_4$, as shown in Fig. 9.7*d*. *The reference frame on the induction motor thereby rotates with the velocity of the rotor $v_4$ in both stator and rotor.*

All base frequencies $f_1$ and $f_4$ become slip frequencies, so that $f_1 = f_4 = 1 - v_4 = s$. The final equivalent circuit is given in Fig. 9.7*e*, and the corresponding reference frames in Fig. 9.7*f*.

The torque of each motor is found, as usual, by

$$T_1 = i^{r1*}B_1 \qquad \text{and} \qquad T_4 = i^{r4*}B_4$$

It should be noted that in the Scherbius rotor the indicated voltage represents $B_1 v_1/s$ and not $B_1$. Hence the measured difference of potential is multiplied by $s/v_1$.

## THE VARIABLE PHASE SHIFTERS OF UNBALANCED MACHINES

In a backward **b** mesh the variable phase shifter assumes an angle $\theta$ having the opposite sign of that taken up by it in the forward **f** mesh, since $\epsilon^{-j\theta}$ represents rotation in the opposite direction from $\epsilon^{j\theta}$.

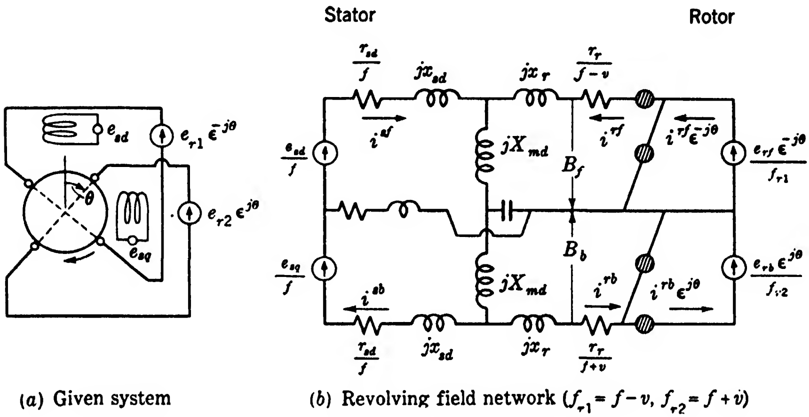



FIG. 9.8. Unbalanced induction motor with reference frames rotating with the rotor.

The more general equivalent circuit for an unbalanced induction motor, whose rotor reference frame rotates with the rotor (but its stator reference frame is still stationary), is given in Fig. 9.8. For a salient-pole synchronous machine the more general circuit is shown in Fig. 9.9. On

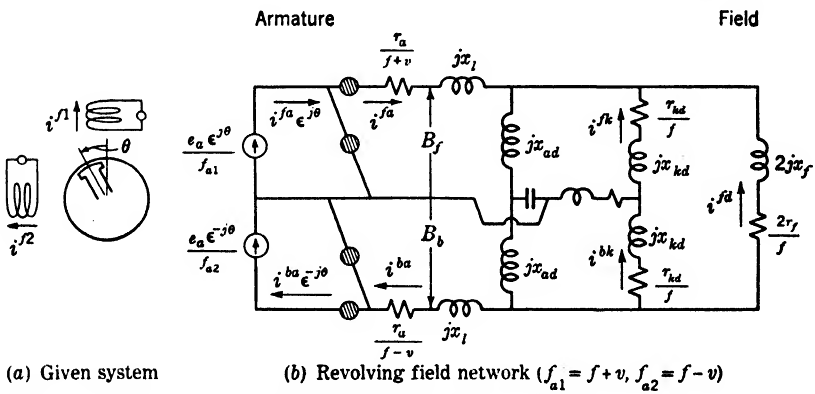


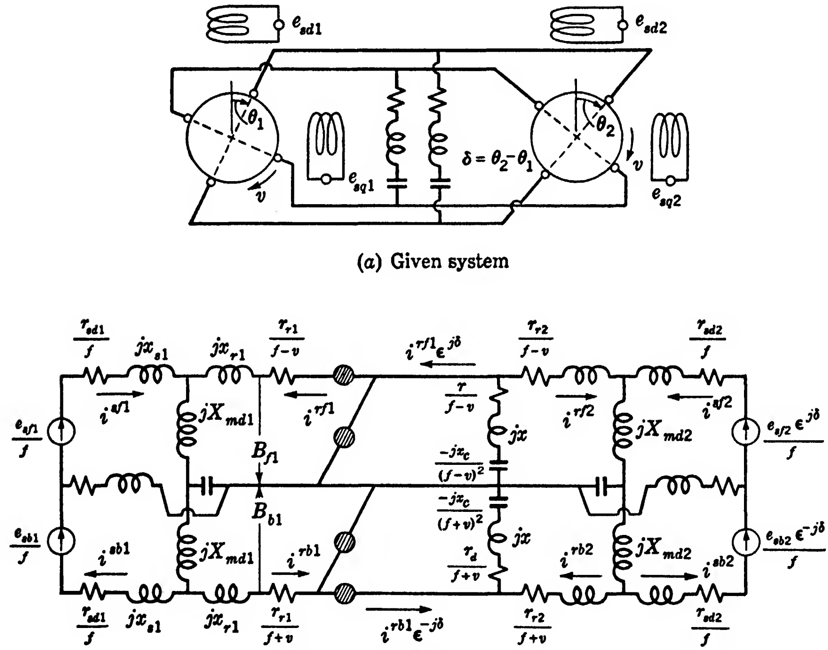

FIG. 9.9. Synchronous machine with reference frames stationary on the armature.

the armature the axes are stationary in space; on the rotor they rotate with the field.

In interconnected machines the forward and backward variable phase shifters are moved about either singly or together, until they combine with each other or are shifted out of the network entirely. It has to be remembered that *in the presence of an unbalanced impedance the constant-angle phase shifters cannot be removed from the equivalent circuit,* as was shown in Chapter 3.

## INTERCONNECTION OF TWO UNBALANCED INDUCTION MOTORS WITH BALANCED LOADS

When two induction motors having unbalanced stator windings run at the same speed and have balanced loads connected across their rotors




(*a*) Given system

(*b*) Revolving-field network

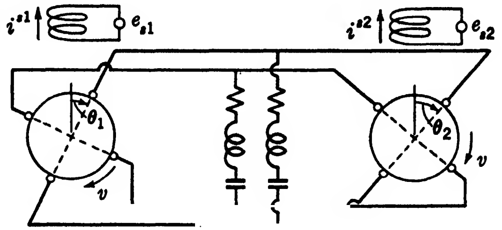
FIG. 9.10. Two unbalanced induction motors with balanced load.

(Fig. 9.10*a*), the steps of Fig. 9.4*b*, *c*, and *d* are repeated for their equivalent circuits also. In the **b** meshes all $\theta$'s and $v$'s merely assume opposite signs. The combined phase shifter with the constant angle $\delta = \theta_2 - \theta_1$ of Fig. 9.10*b* can be moved out to the left only if the stator windings of the first motor are balanced.

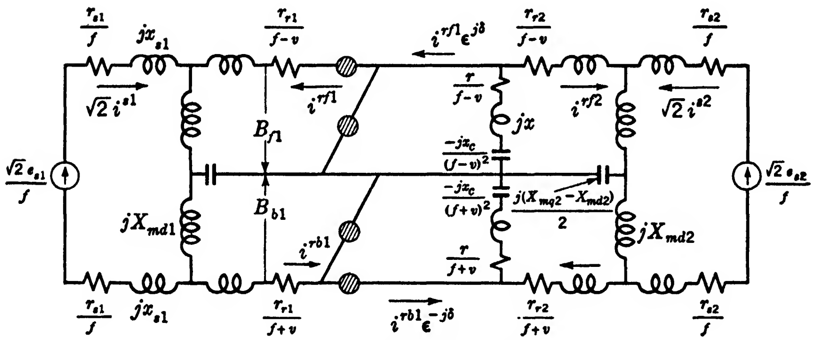
If the load connected across the rotors is unbalanced, the corresponding equivalent circuit is developed in Chapter 11.

## INTERCONNECTION OF TWO SINGLE-PHASE SELSYNS

A special case of the two unbalanced induction motors is two single-phase selsyns, that is, two induction motors in which the stator quadrature-axis windings are missing (or are open-circuited) (Fig. 9.11*a*).




(*a*) Given system




(*b*) Revolving-field network

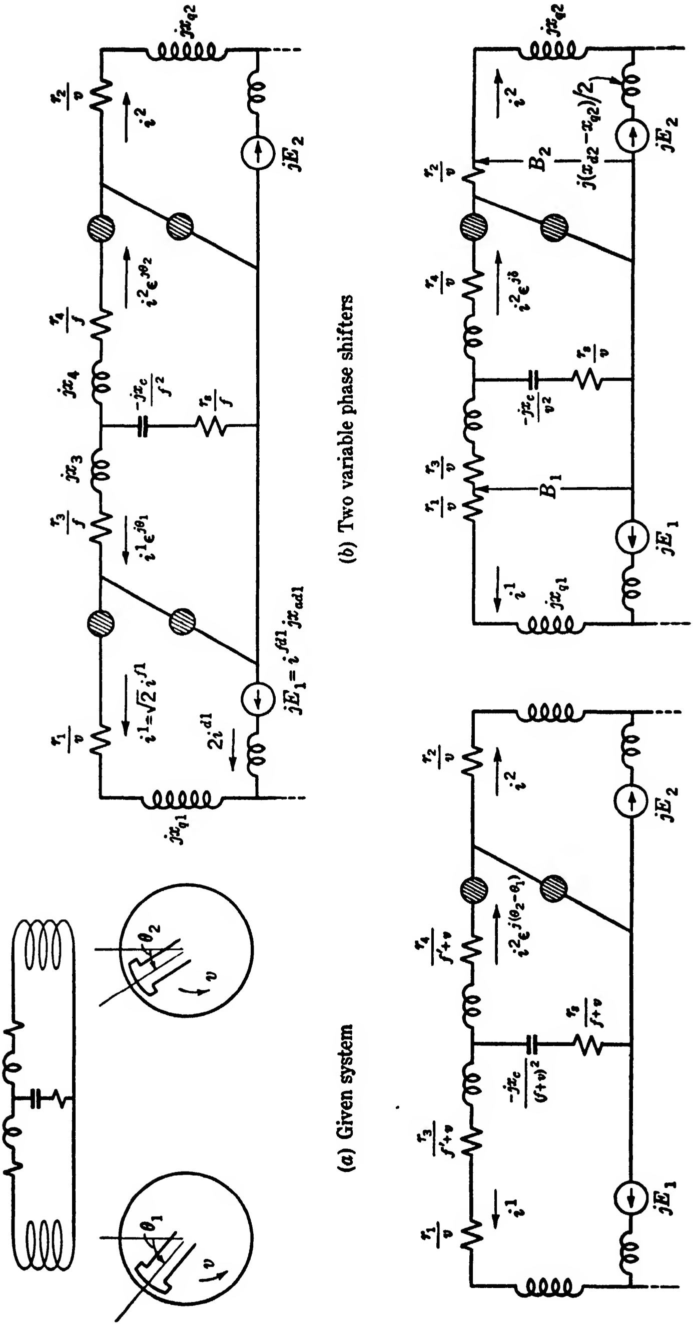
FIG. 9.11. Two single-phase selsyns with unbalanced shunt load.

Since $i^{qs} = 0$ on both machines, *the common stator branches* on Fig. 9.10*b* (in which $ji^{qs}$ flows) *become open-circuited,* as shown in Fig. 9.11*b*.

Here it is assumed that the two single-phase machines run at the same speed. The corresponding equivalent circuits when the machines run at different speeds are treated in Chapter 11.

## INTERCONNECTION OF TWO SALIENT-POLE SYNCHRONOUS MACHINES

It should be recalled from Chapter 6 that the backward **b** meshes of salient-pole synchronous machines may be left out, provided that the excitation voltage $i^{fd}jx_{ad} = jE$ in the common branch is an imaginary number (or the field current $i^{fd}$ a real number). In that event the backward-mesh quantities are the conjugates of the forward-mesh quantities.




(a) Given system

(b) Two variable phase shifters

(c) One constant phase shifter $f' = f - v$

(d) Replacing $f' = 0$ and $\theta_2 - \theta_1 = \delta$

FIG. 9.12. Two synchronous machines with balanced loads.

Let two machines running at the same speed $v$ be connected through a transmission line or a load as shown in Fig. 9.12$b$. The two phase shifters are combined into one with $\delta = \theta_2 - \theta_1$. Because of the saliency, the constant phase shifter cannot be pushed out of the network. The transmission line impedances assume their values viewed from a rotating reference frame (Fig. 9.12$c$). Since the field excitation is usually d-c, the base frequency $f$ is zero. The resultant system is shown in Fig. 9.12$d$.

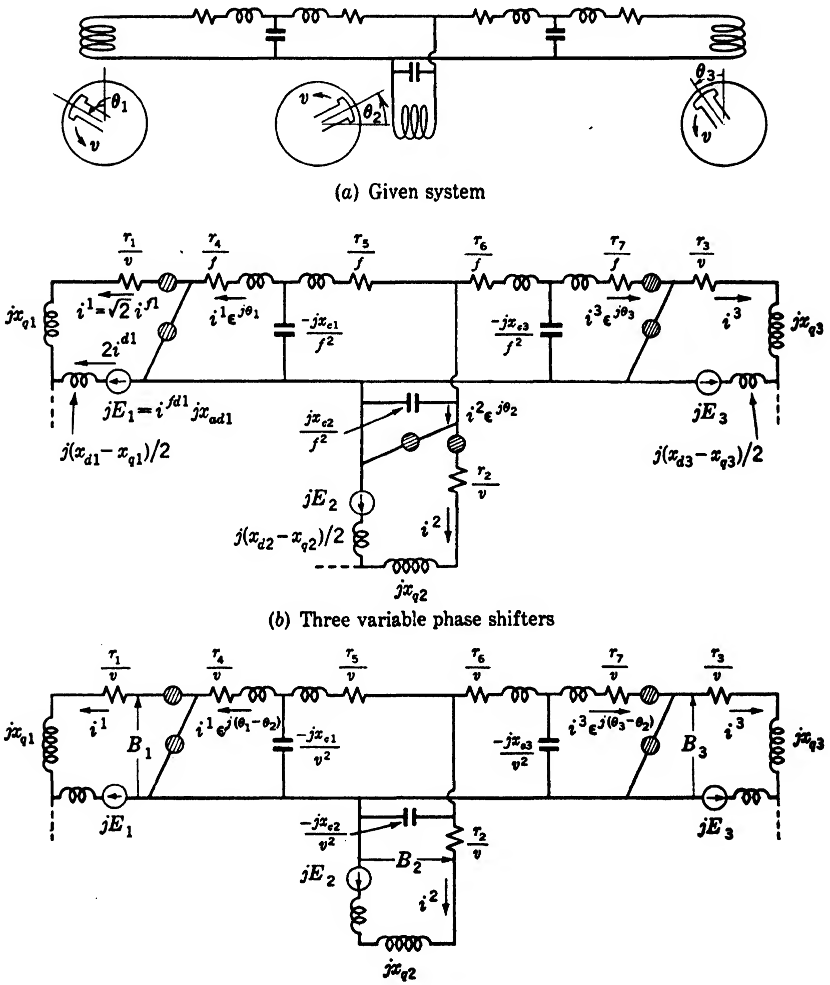
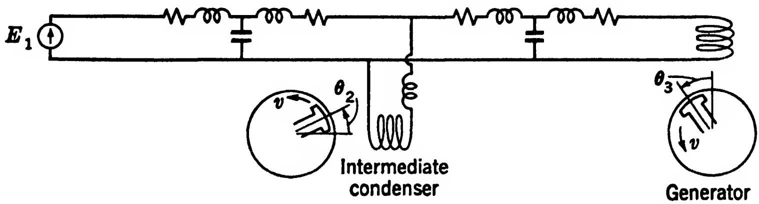


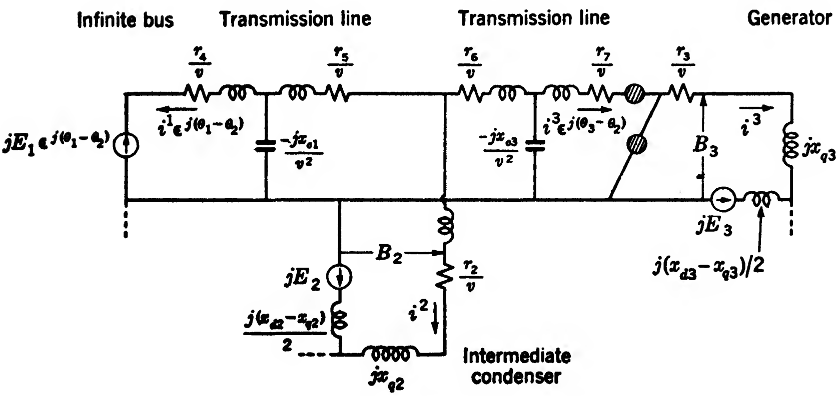

FIG. 9.13. Three synchronous machines with balanced loads.

## INTERCONNECTION OF THREE SALIENT-POLE SYNCHRONOUS MACHINES

When three machines are interconnected by transmission lines (Fig. 9.13), care has to be taken in moving the phase shifter. *When the phase shifter arrives after its journey to a fork, each branch of the fork acquires the same phase shifter.* When phase shifter No. 2 is moved beyond the fork, the three phase shifters of Fig. 9.13*b* are modified to the two constant-angle phase shifters of Fig. 9.13*c* with angles of $\theta_1 - \theta_2$ and $\theta_3 - \theta_2$, respectively. Whereas $\theta_1$, $\theta_2$, and $\theta_3$ are variable angles, $\theta_1 - \theta_2$ and $\theta_3 - \theta_2$ are two independent constant angles, representing the constant-angular displacement between the three machines.




(*a*) Given system




(*b*) One constant phase shifter

FIG. 9.14. Long-distance transmission system with one intermediate condenser.

## LONG-DISTANCE TRANSMISSION SYSTEM

When one of the machines is an infinite bus (Fig. 9.14), one of the constant phase shifters may be moved through it. The synchronous machine in the middle of the line may be a synchronous condenser.

The remaining single phase shifter cannot be eliminated, since it must

split in its journey into two phase shifters, and since one of the two becomes stuck again in machine No. 2.

## TWO SETS OF INDEPENDENT NETWORKS

In all examples of interconnected machines hitherto considered, only one current flowed in each winding, no matter how many voltages were impressed throughout the system. That is, in each mesh the frequency of the impressed voltages was the same as the frequency of current flowing in that mesh. When machines run at different speeds, it often happens that the *frequencies of currents and impressed voltages in the same winding are different.* In this event an independent set of equivalent circuits belongs to each impressed voltage. *There are as many electrically independent networks as there are impressed voltages.*

## SUPERFLUOUSNESS OF THE VARIABLE PHASE SHIFTERS

It should be recalled that the effects of moving the variable phase shifters about are to leave the absolute frequencies $n$ unchanged and to vary the base frequencies $f$. Expressed in another way, when a variable phase shifter separates a mesh into two parts, *the absolute frequencies of the mesh on each side of the phase shifter are identical,* even though they are expressed in terms of different base frequencies.

Hence, as long as the base frequencies are correctly labeled, *machines may be interconnected without the temporary introduction of variable phase shifters.* The only precaution to be observed is that in the meshes to be interconnected the absolute frequencies should be identical.

## MACHINES RUNNING AT DIFFERENT SPEEDS

The auxiliary device of variable phase shifters is particularly useful when their resultant becomes a constant-angle phase shifter. When one or more of the base frequencies are unknown (as in machines running at different speeds), variable phase shifters should be dispensed with entirely and reliance based upon the equality of absolute frequencies for establishing equivalent circuits, as will be shown in the following examples.

## TWO POLYPHASE INDUCTION MOTORS RUNNING AT DIFFERENT SPEEDS

Let two polyphase induction motors running at different speeds be interconnected through their slip rings, with a load across the rings (Fig. 9.15*a*). Let it first be assumed that only one of the motors is excited by a voltage with frequency $f_1$. The equivalent circuit of the three component systems *before* interconnection is shown in Fig. 9.15*b*.

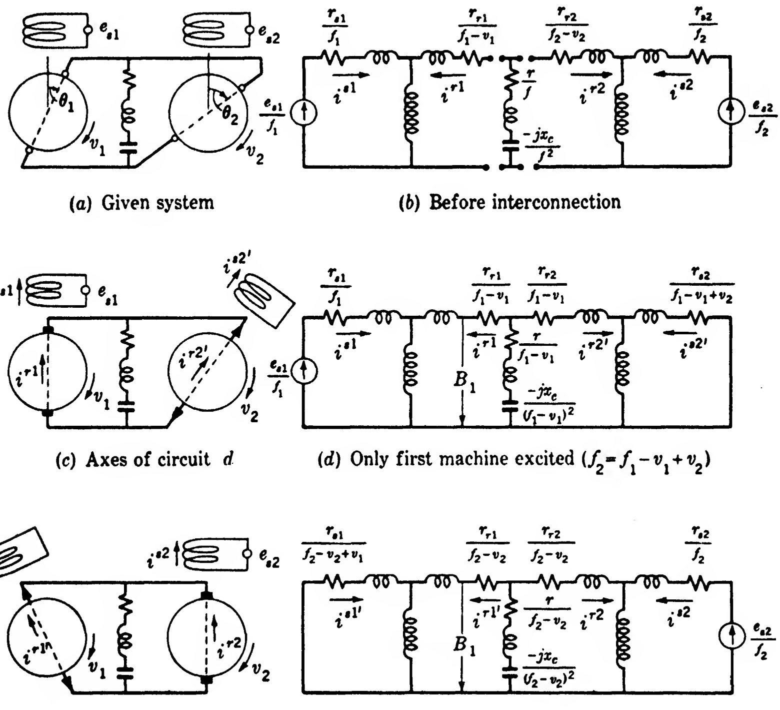
In order that the two rotors could be interconnected, their absolute frequencies must be identical. That is, the following relation must be satisfied:

$$f_1 - v_1 = f_2 - v_2$$

Solving for the unknown base frequency $f_2$ of the second machine,

$$f_2 = f_1 - v_1 + v_2$$

Replacing $f_2$ by this value, the three component systems may be interconnected as shown in Fig. 9.15*d*. The absolute frequency $f$ of the



(*a*) Given system (*b*) Before interconnection

(*c*) Axes of circuit *d* (*d*) Only first machine excited ($f_2 = f_1 - v_1 + v_2$)

(*e*) Axes of circuit *f* (*f*) Only second machine excited ($f_1 = f_2 - v_2 + v_1$)

FIG. 9.15. Two polyphase induction-motors running at different speeds.

stationary network assumes the same value as those of the two rotors, namely, $f_1 - v_1$ (or $f_2 - v_2$). This circuit is the equivalent of a stationary reference frame on the first rotor, as shown in Fig. 9.15*c*.

Assuming an excitation on the second motor, the roles of $f_2$ and $f_1$, also of $v_2$ and $v_1$, are interchanged. The resultant circuit is shown in Fig. 9.15*f*.

The torques of various frequencies due to the cross products of currents and torques are found by means of Eqs. 4.9 and 4.10.

## UNBALANCED MACHINES RUNNING AT DIFFERENT SPEEDS

When the previous two induction motors have unbalanced stator windings, the backward rotor meshes **b** are also interconnected. Now *two sets of frequencies must be equated,* one set for the forward meshes,

$$\begin{aligned} f_1 - v_1 &= f_2 - v_2 \\ f_2 &= f_1 - v_1 + v_2 \end{aligned} \qquad 9.1$$

and another set for the backward meshes,

$$\begin{aligned} f_1 + v_1 &= f_2 + v_2 \\ f_2 &= f_1 + v_1 - v_2 \end{aligned} \qquad 9.2$$

When $v_2$ differs from $v_1$, the base frequency $f_2$ of the second machine assumes two different values in the **f** and **b** meshes, indicating that an infinite series of time harmonics arises. Such an analysis is undertaken in Chapter 11.

If, however, the second machine has balanced windings and a smooth airgap, whereas the first is unbalanced, and if the load between them is balanced, then **b** meshes of the second machine are isolated from its **f** meshes and *the base frequency $f_2$ may assume different values in the* **f** *and* **b** *meshes.*

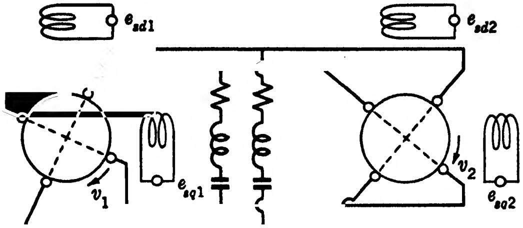
Hence, when two unbalanced machines run at different speeds, an *exact* analysis requires the introduction of time harmonics. However, it is possible to ignore the higher time harmonics and to assume that *as a first approximation the second (unexcited) machine has balanced polyphase windings and a smooth airgap.* Since usually each machine is excited, two independent equivalent circuits are established.

Since each machine is assumed once an unbalanced machine and once a balanced machine, some average is reached between the two extremes. The load between them must, however, be balanced in each case. The impressed voltages alone may be unbalanced for each machine.

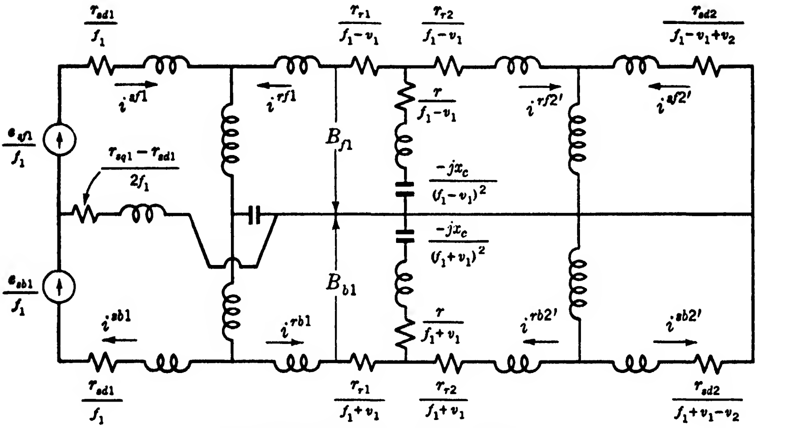
## TWO UNBALANCED INDUCTION MOTORS RUNNING AT DIFFERENT SPEEDS (FIRST APPROXIMATION)

Let two induction motors with *unbalanced* windings and impressed voltages run at different speeds and let a *balanced* load be connected across their rotors (Fig. 9.16*a*).

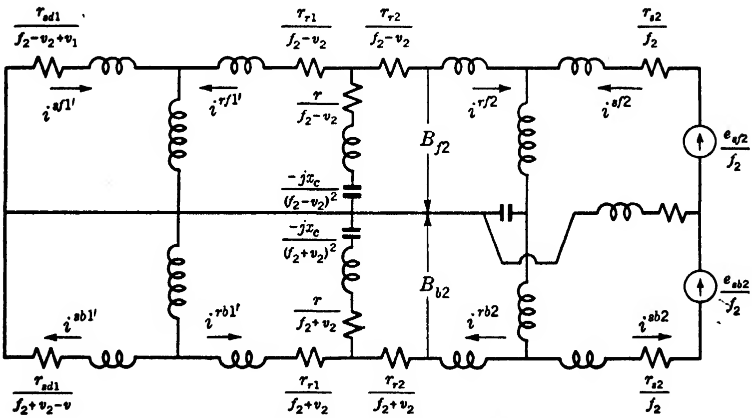
As a first approximation a voltage is assumed to be impressed only on the first motor and the second motor is assumed to be balanced. Equat-




(a) Given system




(b) Only first machine excited ($f_2 = f_1 - v_1 + v_2$)




(c) Only second machine excited ($f_1 = f_2 - v_2 + v_1$)

FIG. 9.16. Two unbalanced induction motors running at different speeds (first approximation).

ing the frequencies of the forward rotor meshes from Eq. 9.1,

$$f_2 = f_1 - v_1 + v_2$$

representing the frequency in the stator **f** mesh of the second machine.

Equating next the frequencies of the backward rotor meshes, from Eq. 9.2,

$$f_2 = f_1 + v_1 - v_2$$

representing the frequency in the stator **b** mesh of the second machine. The resultant circuit is shown in Fig. 9.16*b*.

Assuming now a voltage impressed on the second machine only, with the first machine smooth, the roles of $v_1$ and $v_2$, also $f_1$ and $f_2$, are interchanged. The resultant circuit is shown in Fig. 9.16*c*.

## SINGLE-PHASE SELSYNS OUT OF SYNCHRONISM (FIRST APPROXIMATION)

When the quadrature-axis windings on the stator of two induction motors are missing (or are open-circuited) (Fig. 9.17*a*), the common branch in the equivalent circuit of the stator winding with excitation is left out (Fig. 9.17*b*). *The other stator still is assumed to have a balanced polyphase winding.* Each excitation requires a separate equivalent circuit.

## TWO INTERCONNECTED SALIENT-POLE SYNCHRONOUS MACHINES RUNNING AT DIFFERENT SPEEDS (FIRST APPROXIMATION)

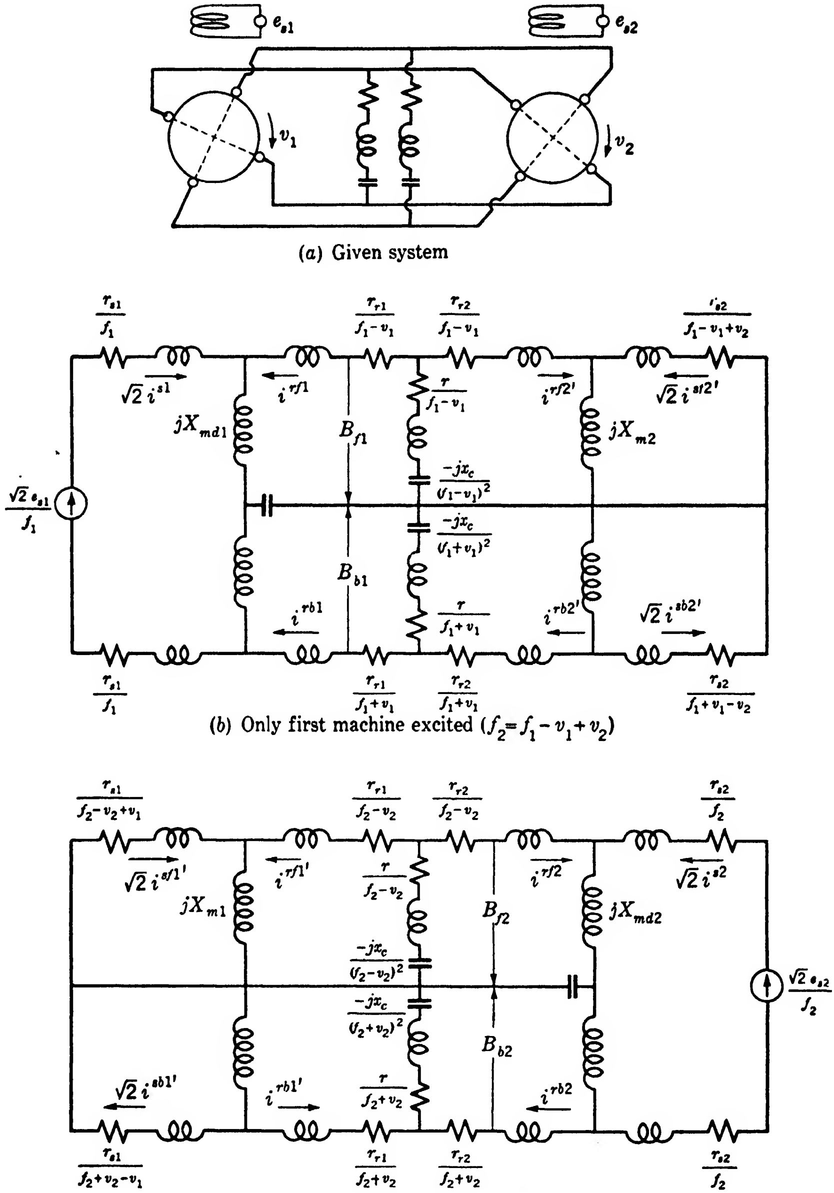
Synchronous machines (Fig. 9.18) differ from single-phase selsyns (Fig. 9.17) in the following respect:

1. The base frequency $f$ is zero.
2. The signs of $v$ are interchanged.
3. The synchronous machines have amortisseur windings.
4. The field mesh, in which the d-c excitation occurs, is eliminated by impressing $i^{fd} jx_{ad}$ in the common armature mesh.
5. The **b** meshes are eliminated in the armature since all $i^b$ are the conjugates of $i^f$.

The correct equivalent circuit of the system, in which the presence of a polyphase field winding is not assumed, is given in Chapter 11.

## NUMERICAL EXAMPLE OF A WIND-TUNNEL FAN DRIVE

Let the fan drive of Fig. 9.6 be considered and let the synchronous motor running at the slip speed of the induction motor possess saliency. As mentioned at the end of Chapter 6, the saliency effect may be considered by assuming in Fig. 9.6*d* an impedance added to the common

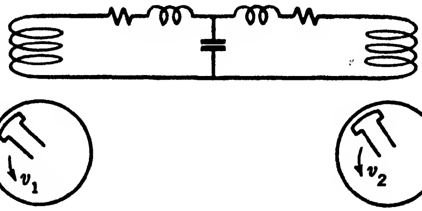


(a) Given system

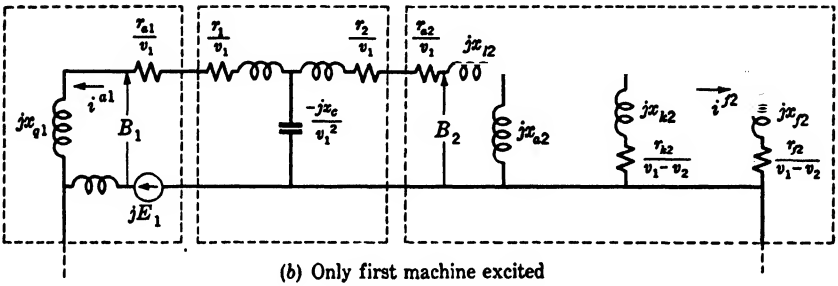
(b) Only first machine excited ($f_2 = f_1 - v_1 + v_2$)

(c) Only second machine excited ($f_1 = f_2 - v_2 + v_1$)

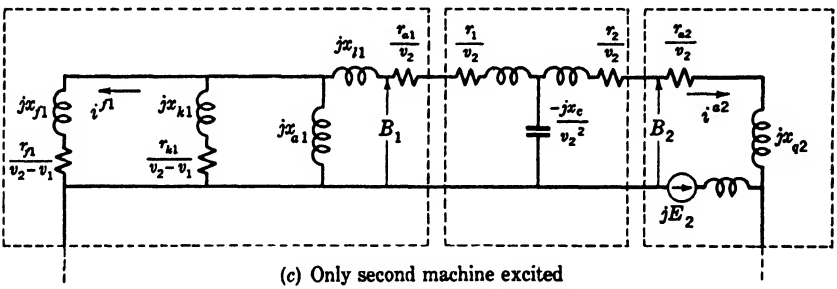
FIG. 9.17. Two single-phase selsyns running at different speeds (first approximation).



(a) Given system



(b) Only first machine excited



(c) Only second machine excited

FIG. 9.18. Two synchronous machines running at different speeds (first approximation).

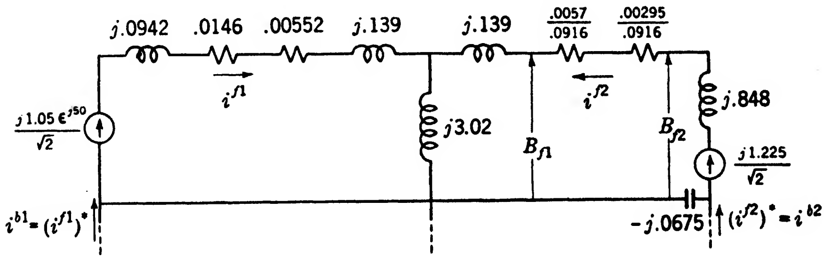



FIG. 9.19. Numerical example of the wind-tunnel fan drive of Fig. 9.6d.

branch, as shown in Fig. 9.19. Since both backward and forward meshes will be used in the calculation, all voltages and currents in Fig. 9.6*d* will be divided by $\sqrt{2}$, as indicated in Fig. 6.4*c*.

The first alternator running at unit speed is assumed to be an infinite bus connected to the induction motor through a tie line, with an impedance $R + jX$. The constants of the system are given in Table 9.1.

TABLE 9.1 CONSTANTS OF A FAN DRIVE

| Infinite Bus | Induction Motor | Synchronous Motor |
|---|---|---|
| $e = 1.05$ | $r_s = 0.00552$ | $r_s = 0.00295$ |
| Tie line | $x_s = 0.139$ | $x_d = 0.983$ |
| $R = 0.0146$ | $x_m = 3.02$ | $x_q = 0.848$ |
| $x = 0.0942$ | $x_r = 0.139$ | $E = 1.225$ |
| | $r_r = 0.0057$ | |

Let the angle $\delta$ at some particular load on the induction motor be $-50°$ (at no load $\delta = 0$) (also let the synchronous motor excitation be $i^{fd}x_{ad} = E = 1.225$). Let the rotor of the synchronous motor be driven at a slip $s = v_3 = 0.0916$. The problem is to find the currents and torques of the two machines.

To find the two forward currents, the voltage equations of the two forward meshes of Fig. 9.19 are set up. The impressed voltage on the first mesh is

$$\frac{j1.05(\cos 50 + j\sin 50)}{\sqrt{2}} = -0.569 + j0.477$$

and that on the second mesh is $j0.866$. The voltage equations of the two meshes are (assuming all currents as flowing in closed meshes)

$$-0.569 + j0.477 = (0.0201 + j3.25)i^{f1} + j3.02i^{f2}$$

$$j0.866 = j3.02i^{f1} + (0.0944 + j4.0745)i^{f2} - j0.0675(i^{f2})^*$$

In these two *complex* equations apparently there are three complex unknowns, $i^{f1}$, $i^{f2}$, and $(i^{f2})^*$. However, if the equations are rewritten as four *real* equations, the number of real unknowns is four.

$$i^{f1} = -0.217 + j0.5125$$

$$i^{f2} = 0.39 - j0.409$$

Since we know the currents, the two fluxes are

$$B_{f1} = 0.383 - j0.523$$

$$B_{f2} = 0.3467 + j0.484$$

The torques are, by $T = 2i^{f*}B_f$,

$$T_1 = 0.726 \qquad \text{and} \qquad T_2 = 0.665$$

# 10 SPACE HARMONICS

## SPACE HARMONICS AS SEPARATE MACHINES

All machines hitherto considered have been assumed to have *two poles* only. That is, all electromagnetic waves (current densities and flux densities) had only one pair of poles. Although in salient-pole machines the *permeance* wave around the airgap has two pairs of poles, the higher space harmonic fluxes due to it were ignored.

However, the distribution of currents in slots, the presence of salient poles, etc., introduce higher space harmonics of current density and flux density waves in each winding. To establish equivalent circuits for their study, it will be assumed that *each particular set of space harmonics exists in a separate machine, and so as many separate machines of the same type have to be interconnected as there are space harmonics to be considered.* These machines with identical structures differ, however, in the following respects:

1. Each has a different number of pairs of poles $P$.
2. They are interconnected through either their stator windings or rotor windings, or both.
3. Their design constants differ either by a harmonic pitch factor or by some other factor.

Hence *the problem of space harmonics consists of the interconnection of several similar machines running at the same speed and having a different number of pairs of poles.* The calculation of the harmonic pitch factors is not the subject matter of the present volume. The calculation may be found elsewhere.*

## MACHINES WITH $P$ PAIRS OF POLES

The revolving-field and cross-field equivalent circuits of any rotating machine with $P$ pairs of poles are the same as for one pair of poles, with the difference that *in the absolute frequency expressions the rotor speed $v$*

* G. Kron, *Tensor Analysis of Networks*, Chapter XII, "Reactance Calculation of Windings," p. 296, John Wiley & Sons, New York, 1939.

*is replaced by Pv*, where $v$ represents the number of *electrical* radians described per second. That is, in such cases

$$v = \frac{\text{Actual rpm}}{\text{Two-pole synchronous rpm}}$$

For instance, 900 rpm would correspond to $v = 0.25$ for a two-pole machine (on 60 cycles), $v = 0.5$ for a four-pole machine ($P = 2$), and $v = 1$ for an eight-pole machine ($P = 4$).

## THREE INDUCTION MOTORS IN SERIES

As an example, let the *stator* windings of three polyphase induction motors with $P_1$, $P_2$, and $P_3$ pairs of poles be interconnected *in series.*

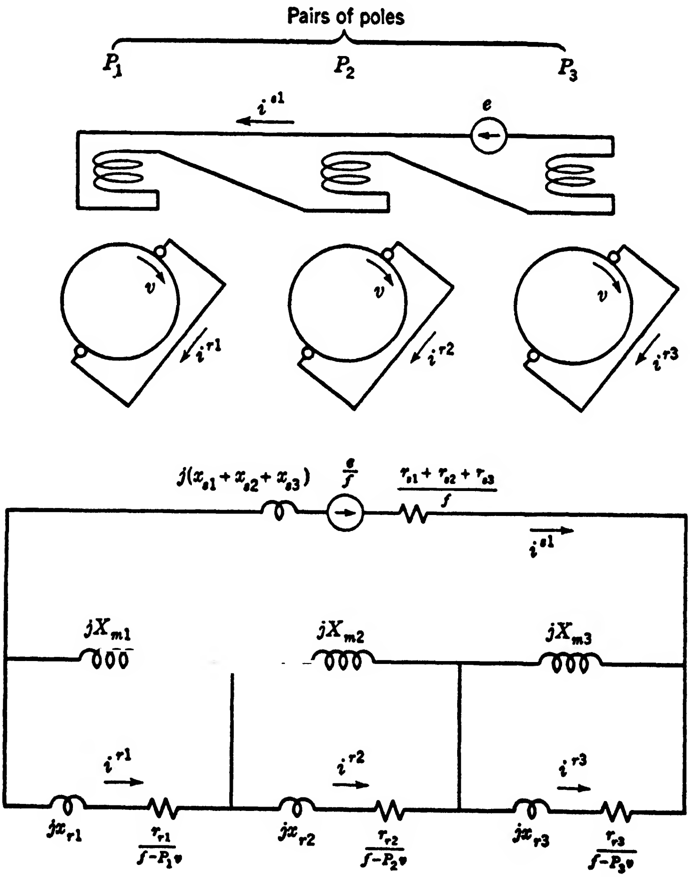



FIG. 10.1. Three induction motors in series (each with different number of poles).

Also let their rotors be coupled together to run at the same speed $v$. Their equivalent circuit is shown in Fig. 10.1. The absolute frequency in each of the rotor windings is different.

When their *rotor* windings are connected in series (Fig. 10.2), then the absolute frequencies in their stator windings are different. *In going from a stator to a rotor mesh (and vice-versa) the absolute frequency decreases by Pv instead of v.*

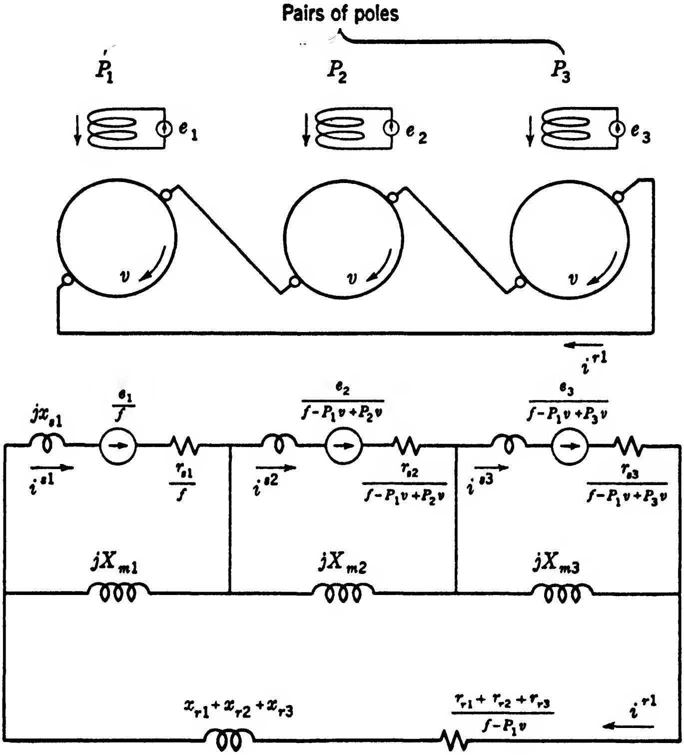



FIG. 10.2. Three induction motors in cascade (each with different number of poles).

## EXAMPLES TO BE CONSIDERED

Two detailed examples of space harmonics will be considered:

1. The space harmonics of polyphase induction motors are the most thoroughly investigated subject on harmonics, because of their importance in the study of iron losses, locking torques, hooks in the speed-torque curves, vibration, and noise.

At first the harmonics produced by a constant airgap and harmonic

mmf waves are studied; afterward the additional harmonics due to the stator and rotor slot openings are introduced. *The multichain equivalent circuits to be derived are easily solvable by hand, as each mesh to be eliminated has only one neighbor.* (The a-c board may also be used.) As special cases of slot openings, an eccentrically spaced rotor and a bent rotor are also considered.

2. The space harmonics of the shaded-pole motor are quite large compared with the fundamental. No study is acceptable that does not consider the influence of several of the space harmonics produced by the stator windings.

The space-harmonic equivalent circuits of the shaded-pole motor reduce to those of balanced or unbalanced induction and synchronous machines as special cases.

## HARMONICS OF A POLYPHASE INDUCTION MOTOR

We shall consider a polyphase induction motor first and assume that the airgap is uniform and that any harmonic wave is due to the concentration of windings in slots. The sequence of harmonic production is as follows:

1. A generated voltage wave produces a current wave.
2. A current produces an mmf wave.
3. An mmf produces a flux wave.
4. A flux produces a generated voltage wave.

For the construction of an equivalent circuit it is sufficient to consider the production of currents and fluxes only (steps 2 and 4), since in the equivalent circuit they appear as currents and voltages (differences of potential), respectively. The intermediary steps of mmf and generated voltage production will be left out, though understood to exist.

Two attributes of each harmonic wave must constantly be kept in mind, namely, the number of pairs of poles (p.p.) and frequency.

## PAIRS OF POLES

The concept of pairs of poles will enter whenever a flux is created by a current. The flux will lie on the same member as the current and it will have the same frequency as the current producing it, but its number of pairs of poles will be different. In particular:

(*a*) A new flux appearing on the *rotor* (owing to a current flowing in the rotor) will have the following pairs of poles:

$$(\text{p.p. of } \phi_r) = (\text{p.p. of } I_r) + kR \qquad 10.1$$

where $R$ is the number of pole-phase-groups of the rotor winding (or the number of rotor slots) and $k$ is any positive or negative integer, includ-

ing zero. In a squirrel-cage rotor the number of pole-phase-groups is the same as the number of rotor slots.

(*b*) A new flux appearing on the *stator* (owing to a current on the stator) will have the following pairs of poles:

$$(\text{p.p. of } \phi_s) = (\text{p.p. of } I_s) + kQ \qquad 10.2$$

where $Q$ is the number of pole-phase-groups of the stator winding.

## THE "ABSOLUTE" FREQUENCY

Considering next the reverse process, namely, the creation of a current by a flux, the current and the flux will lie on different members. A stator flux will produce a rotor current, and a rotor flux will create a stator current. The number of pairs of poles of the current will be the same as that of the flux. The actual frequency also will be the same, provided that the frequency of the current is measured with respect to the same reference frame as that of the flux. The "absolute" frequencies needed for the equivalent circuits are determined as follows:

(*a*) A new current appearing on the *rotor* due to a flux coming from the stators has the following absolute frequency with respect to the rotor:

$$(\text{Frequency of } I_r) = (\text{Frequency of } \phi_s) - (\text{p.p. of } \phi_s)v \qquad 10.3$$

where $v$ is the speed of the rotor expressed as a fraction of its *two-pole* synchronous speed (3600 on 60 cycles).

(*b*) A new current appearing on the *stator* due to a flux coming from the rotor has the following absolute frequency with respect to the stator:

$$(\text{Frequency of } I_s) = (\text{Frequency of } \phi_r) + (\text{p.p. of } \phi_r)v \qquad 10.4$$

Note the change in sign in the last two formulas.

## THE SEQUENCE OF HARMONIC PRODUCTION

The sequence of production of the harmonic currents and fluxes is as follows:

1. First stator group.

(*a*) A fundamental frequency $f$ current flows in the stator wound with $P$ pairs of poles,

$$I_s = I_s \cos (Px - ft)$$

(*b*) If the number of pole-phase groups of the stator winding is $Q$, the stator fluxes produced by the above current have the same frequency $f$, but the pairs of poles are increased by $k_1Q$ (Eq. 10.2),

$$\phi_s = \phi_s \cos [(P + k_1Q)x - ft]$$

2. First rotor group.

(*a*) Each of these fluxes produces in the rotor a current-density wave having the same number of pairs of poles and the same frequency.

For the equivalent circuit it will be necessary to express the frequency of each rotor current with respect to the rotating conductor, instead of a stationary frame. This "absolute" frequency is the old frequency *diminished* by $v$ times the pairs of poles (Eq. 10.3) so that

$$I_r = I_r \cos \{(P + k_1Q)x - [f - (P + k_1Q)v]t\}$$

where $v$ is the two-pole synchronous speed of the rotor.

(*b*) If the number of squirrel-cage bars (or the number of pole-phase-groups of the rotor winding) is $R$, then the rotor fluxes produced by each of these rotor currents have the same frequency and the pairs of poles are increased by $k_2R$ (Eq. 10.1).

$$\phi_r = \phi_r \cos \{(P + k_1Q + k_2R)x - [f - (P + k_1Q)v]t\}$$

The frequency is with respect to the rotating rotor conductors.

3. Second stator group.

(*a*) Each of these rotor fluxes produces in the stator a current-density wave having the same number of pairs of poles and the same frequency.

For the equivalent circuit the frequency of the stator current must be expressed with respect to the stationary stator conductors. This new frequency is the old frequency *increased* by $v$ times the pairs of poles (Eq. 10.4)

$$[f - (P + k_1Q)v] + (P + k_1Q + k_2R)v = f + k_2Rv$$

The current-density wave on the stator is then

$$I_s = I_s \cos [(P + k_1Q + k_2R)x - (f + k_2Rv)t]$$

(*b*) Each of these currents produces a series of stator fluxes having the same frequency, but the pairs of poles are increased by $k_3Q$.

$$\phi_s = \phi_s \cos \{[P + (k_1 + k_3)Q + k_2R]x - (f - k_2Rv)t\}$$

4. Second stator group.

(*a*) Each stator flux produces one rotor current.

(*b*) Each new rotor current produces a series of rotor fluxes.

The groups may be continued indefinitely.

## SUMMARY

1. The possible pairs of poles are $P + k_1Q + k_2R$.

2. The possible absolute frequencies on the stator (with respect to the stationary stator conductors) are $f + kRv$.

3. The possible absolute frequencies on the rotor (with respect to the rotating conductors) are $f - (P + kQ)v$.

The speed of the fluxes is found by dividing their frequencies by their pairs of poles. The speed of fluxes plays no part in the following. The pairs of poles are needed only to determine the frequency, whereas the frequencies are the only significant quantities to be needed in the construction of equivalent circuits.

In a squirrel cage a flux with any number of pairs of poles may induce a voltage, but that is not true in a wound structure. In a winding the only flux that may produce a voltage is one that has a number of pairs of poles that the winding itself may produce. For any other flux the winding appears open-circuited; hence *the chain of circuits is broken* at that point.

## A MODEL OF THE SPACE HARMONICS

If it is assumed that the harmonic fluxes having different numbers of pairs of poles do not interfere with each other, they may be assumed to exist in different induction-motor structures. These motors are wound with different numbers of pairs of poles, each containing one set of fundamental stator and rotor current and flux, without harmonics. Each motor has different mutual and leakage inductances, also resistances. (The resistances are different for each motor because of the skin effect due to currents of different frequencies.) All rotors run, however, at the same speed $v$ and in the same direction in which the original motor runs.

The fact that one current produces a series of fluxes is represented on the model by connecting a group of stators or rotors in series. These series-connected structures have the same number of pairs of poles as the fluxes produced by the current.

A model for the first stator and rotor group of harmonics is shown in Fig. 10.3. It is assumed for convenience that on both stator and rotor only two harmonics are produced ($k_1$ and $k_2$ assume only the values $0, +1, -1$). The right-hand third of Fig. 10.3 is repeated on Fig. 10.4 and is continued to include a second stator and rotor group of harmonics.

When two harmonic fluxes have the same number of pairs of poles and the same frequency, they are known to interact and produce subsynchronous crawlings. The model does not take care of such phenomena. However, the model does represent the phenomena when the fluxes produce only induction-motor torques (asynchronous crawling).

The immediate problem is to establish an equivalent circuit for the group of induction motors on Figs. 10.3 and 10.4.

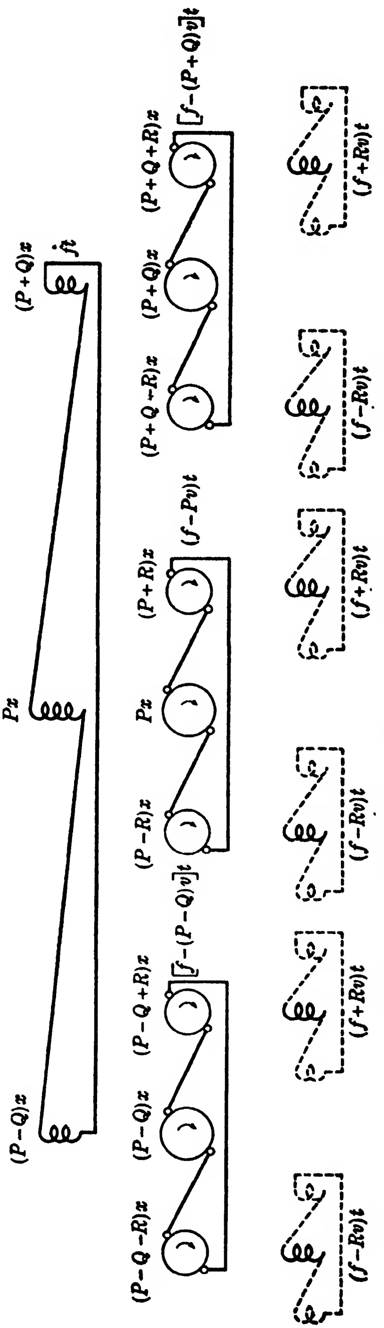



FIG. 10.3. Model of the first stator and first rotor group of harmonics.

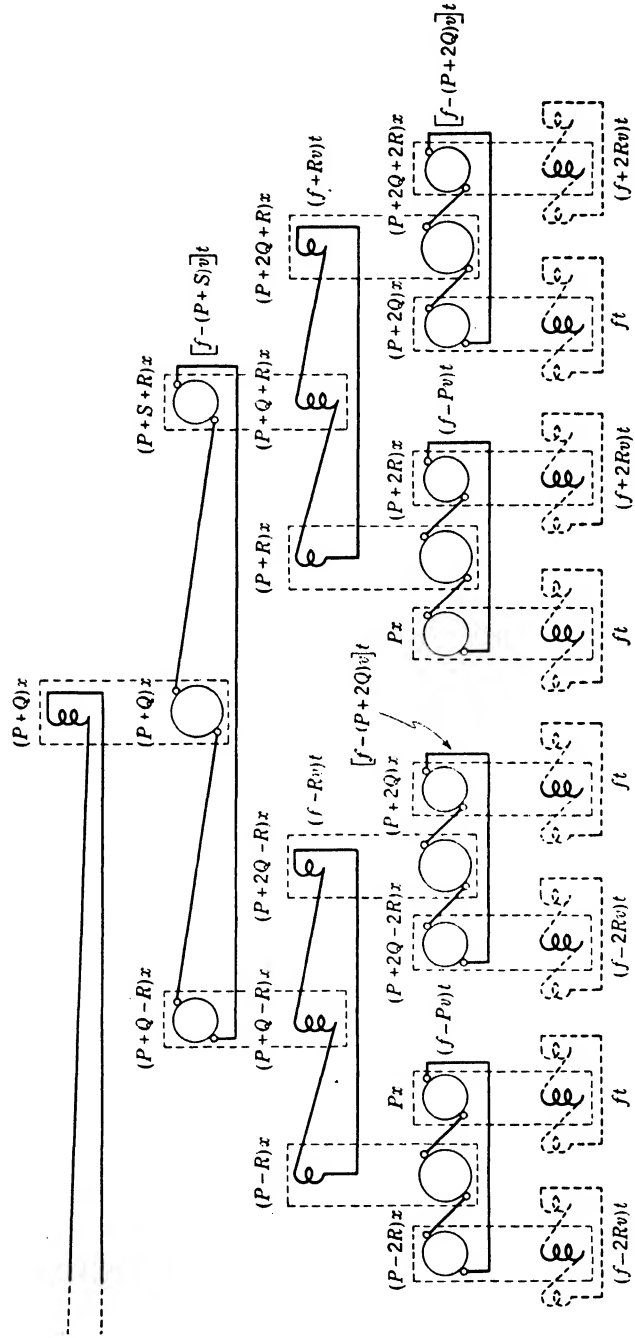



FIG. 10.4. Model of two stator and two rotor group of harmonics due to one stator harmonic. (Continued from Fig. 10.3.)

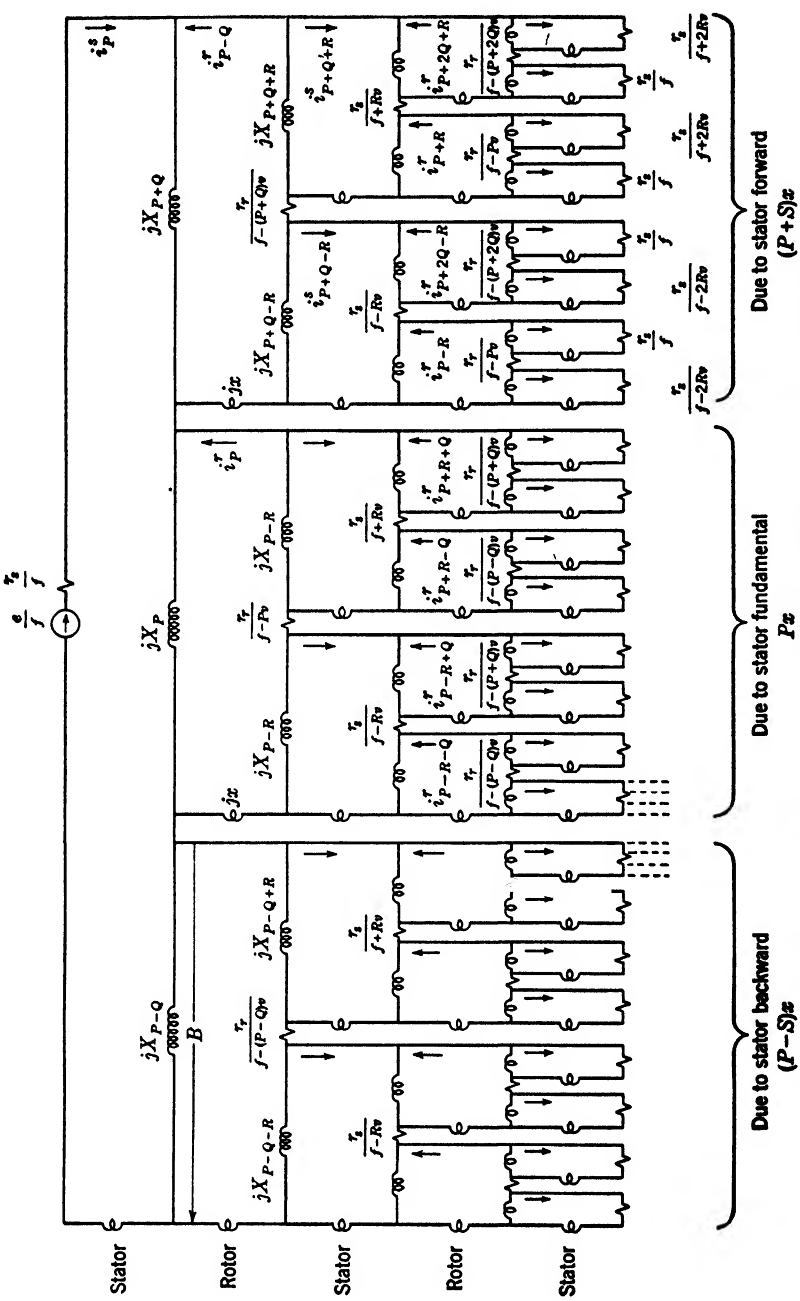



FIG. 10.5. Equivalent circuit of induction-motor harmonic fluxes due to harmonic mmf waves.

## THE EQUIVALENT CIRCUIT

The equivalent circuit of the group of induction motors is established by assuming, first, all rotors stationary and then *connecting the resulting two-winding transformers (each transformer representing a motor) in exactly the same manner as the induction motors are connected.* The resultant network is shown on Fig. 10.5. *The rotation is taken care of by dividing each resistance by the "absolute" frequency of current flowing through that particular resistance.* Impressed emf exists only in the mesh of the stator fundamental current. Neither the speed of fluxes nor their pairs of poles enter into the construction of the network; only their absolute frequency does.

If in the original motor some of the rotor (stator) fluxes do not induce a voltage in the stator (rotor), the corresponding stator (rotor) mesh is simply left *open-circuited.* If some of the fluxes are to be ignored (because of their high number of pairs of poles or some other reason), the corresponding mutual reactances $X_m$ are *short-circuited.*

## THE "RULE OF SPEED" IN CROSSING THE AIRGAP

In the construction of space-harmonic networks it is worthwhile to keep in mind the following rule (Fig. 10.5):

In crossing the airgap from the stator to rotor (and back again), the absolute frequency decreases (increases) by the rotor speed $v$ multiplied by the pairs of poles of the airgap reactance $X$ crossed.

The subscript of each reactance describes the number of pairs of poles.

## RESULTANT MMF'S

The voltage across each horizontal coil in Fig. 10.5 represents an airgap flux. Each flux is caused by a "resultant mmf," represented by the current flowing through the coil.

It should be noted that—just as in the primitive network—the current flowing through each *horizontal* coil has no physical existence in the induction motor. Only the vertical currents (those flowing through a resistance or a leakage inductance) exist in the motor. *The horizontal currents correspond to the resultant mmf's* producing a flux in the presence of the constant airgap permeance. (Compare with Fig. 8.6.)

The number of pairs of poles of each resultant mmf is shown by the subscript of its $X$ or $i$, and its frequency is found simply by observing the two resistances through which the two components of the mmf (the actual currents) flow. The denominators of the two resistances give the frequency of the resultant mmf with respect to the stator and to the rotor, respectively.

As an example, near the center of the third horizontal line on Fig. 10.5

through the coil $X_{P+R}$ flows an mmf with $P + R$ pairs of poles. Its frequency with respect to the rotor is $f - Pv$,

$$\text{Mmf}_s = I \cos [(P + R)x - (f - Pv)t]$$

and with respect to the stator is $f + Rv$,

$$\text{Mmf}_r = I \cos [(P + R)x - (f + Rv)t]$$

This mmf produces with the constant airgap a flux (or the voltage across $jX_{P+R}$) whose pairs of poles and frequencies are the same as those of the mmf.

## FLUXES, TORQUES, LOSSES, AND FORCES

Just as in the standard equivalent circuit, the rms flux density $B$ of each particular harmonic is given by the difference of potential appearing across a mutual reactance $X_m$ (and a leakage reactance $x$). The harmonic torques are found by formulas analogous to Eq. 4.10.

The total losses are found by adding up $i^2r$ losses. The actual resistances must be considered without their denominators.

The harmonic forces are found as products of the various fluxes. The frequencies and number of pairs of poles of the fluxes (hence of the forces) are known from the equivalent circuit. The detailed study of the forces is not undertaken here.

## EFFECTS OF STATOR-SLOT OPENINGS *

The effect of the stator-slot openings ($S$ in number) is to introduce, in addition to the constant airgap permeance $P_c$, a set of sinusoidal permeance waves $P_s = \cos kSx$. With respect to the stator this set of permeance waves is stationary, and with respect to the rotor it rotates backward with speed $v$

$$P_s = P_s \cos (kSx + kSvt)$$

Now, just as each resultant mmf (horizontal current) produces a flux with the constant permeance $P_c$, each resultant mmf will produce with each $P_s$ two fluxes. For instance, the previously mentioned mmf wave,

$$\text{Mmf}_s = I \cos [(P + R)x - (f + Rv)t]$$

and the first-harmonic stator permeance wave, $P_s = P_s \cos Sx$, produce two fluxes:

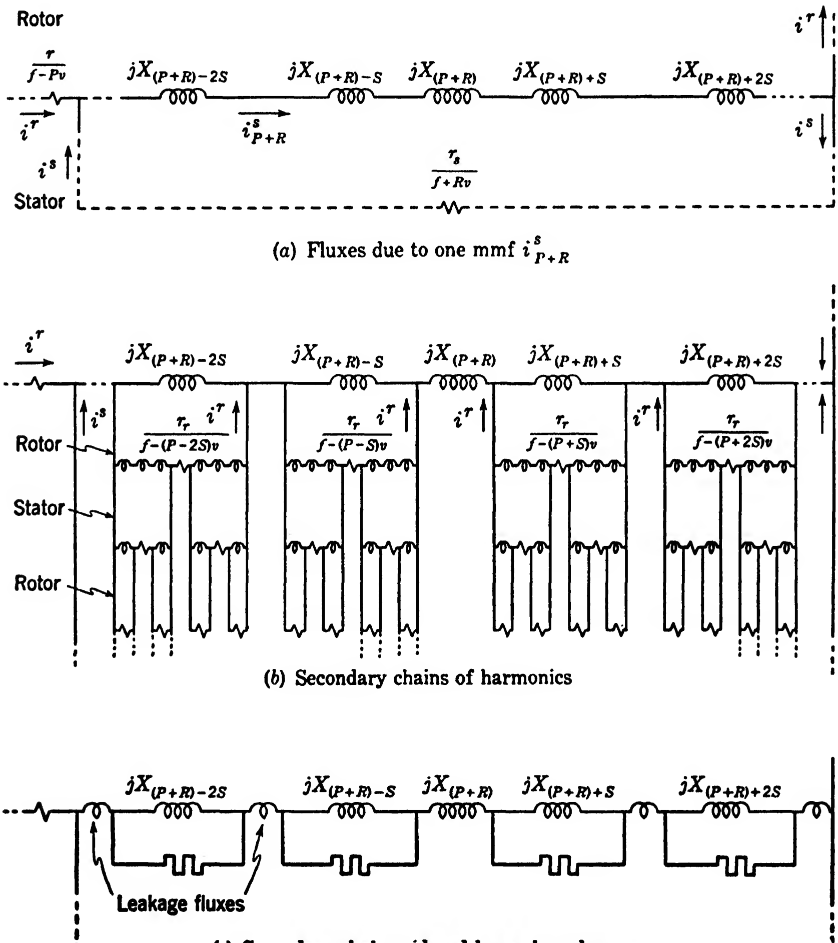
$$\phi_{s1} = \phi_{s1} \cos [(P + R + S)x - (f + Rv)t]$$

$$\phi_{s2} = \phi_{s2} \cos [(P + R - S)x - (f + Rv)t]$$

* Gabriel Kron, "Induction Motor Slot Combinations," *Transactions of the AIEE*, Vol. 50, pp. 757–768, 1931.

These fluxes have the same frequency, when viewed from the stator, as the mmf producing them.

Hence, *as a first step*, the effect of the stator-slot openings is to introduce a double set of inductors in series with each previous horizontal




(*a*) Fluxes due to one mmf $i^s_{P+R}$

(*b*) Secondary chains of harmonics

(*c*) Secondary chain replaced by an impedance

FIG. 10.6. Effects of stator slot openings on an mmf.

inductor (Fig. 10.6*a* shows two inductors for the above-mentioned mmf). The subscript on each previous coil is increased by $kS$, where $k$ is any plus or minus integer. The voltage across each coil is the flux introduced by the particular stator-slot harmonic permeance. The voltages add up, since they all have the same frequency.

### THE FREQUENCY OF SLOT HARMONICS

Since $Q$, the number of pole-phase-groups of the stator winding, is an integer multiple of the stator slots $S$, it will be found that these new fluxes all have the same number of poles as have some of the previous fluxes that lie along the same horizontal line. *The slot openings introduce no new pairs of poles* into a machine with balanced windings.

However, if the frequency of the "slot-opening" fluxes is considered, these frequencies will be found to be of a different type from those of the previous, "constant-airgap" fluxes. In particular, it will be found that *the fluxes due to the stator-slot openings have different frequencies with respect to the rotor from those of the resultant mmf's producing them.* With respect to the stator both mmf's and fluxes have the same frequencies (even though different pairs of poles).

It should be recalled that the constant-airgap fluxes have the same frequency with respect to both stator and rotor as the resultant mmf's producing them.

As an example, the previously mentioned mmf has the following two forms with respect to the rotor and stator, respectively:

$$\mathrm{Mmf}_S = I \cos [(P + R)x - (f - Pv)t]$$

$$\mathrm{Mmf}_R = I \cos [(P + R)x - (f + Rv)t]$$

Considering $\phi_{s1}$, with respect to the stator it is

$$\phi_{s1} = \phi_{s1} \cos [(P + R + S)x - (f + Rv)t]$$

It has the same frequency as the mmf producing it. But with respect to the rotor the same flux is

$$\begin{aligned}\phi_{s1} &= \phi_{s1} \cos \{(P + R + S)x - [(f + Rv) - (P + R + S)v]t\} \\ &= \phi_{s1} \cos \{(P + R + S)x - [f - (P + S)v]t\}\end{aligned}$$

This frequency differs from that of the resultant mmf, which is $f - Pv$.

### THE NEW ROTOR CHAINS OF HARMONICS

When one of these stator-permeance fluxes cuts the stator winding, there exists already an mmf *of the same frequency* to support the flux. However, when such a flux cuts the rotor, there exists no mmf of the same frequency to support the flux. Hence *a new rotor current flows,* having the same pairs of poles and the same frequency as the flux cutting it,

$$I_r = I_r \cos \{(P + R + S)x - [f - (P + S)v]t\}$$

Expressed in another manner, *each permeance flux is produced by three different currents instead of two.* This third current is a rotor current (for the stator-slot-opening flux).

This additional current starts in its wake a new chain of harmonics, as shown in Fig. 10.6*b*. Each of the new secondary chains may be replaced by an effective impedance, as shown in 10.6*c*.

## THE CHAIN IMPEDANCE

The resultant equivalent circuit for one set of stator-slot-opening harmonics ($k = \pm 1$) is shown in Fig. 10.7. It looks the same as Fig. 10.5, with the difference that the original reactances include the effect of the stator-slot openings, in addition to the constant airgap.

Considering the chains started by the coils on *every other* horizontal line (those lying between meshes marked "stator" *above* them and "rotor" *below* them), these chains correspond to already-existing chains along the same horizontal line. The chains starting on the other lines have their correspondence only two levels lower.

It is possible to assume that only part of these secondary harmonic fluxes reaches the rotor and starts a new chain. The remaining part of the secondary fluxes appears as leakage, as shown in Fig. 10.6*c*. If these secondary harmonics are entirely ignored, then the new slot-opening reactances appear merely as leakage reactances, as shown in Fig. 10.6*a*, and may be grouped together with the original constant-airgap reactance.

## ECCENTRIC ROTORS

A special case of a stator-slot permeance is an airgap permeance produced by a rotor placed permanently off center. An airgap permeance with *one* pair of poles and stationary in space appears (as if the stator had *one* large slot). When the fundamental mmf has $P$ pairs of poles $\cos(Px - ft)$, the two fluxes are $\cos[(P+1)x - ft]$ and $\cos[(P-1)x - ft]$. Each flux has a different frequency with respect to the rotors.

When $P$ is assumed to be two or larger, *the two fluxes due to the rotor eccentricity, in general, do not generate voltages in the stator,* but only in the squirrel-cage rotor. Hence in this respect the equivalent circuit must differ from Fig. 10.7.

The effect of the stator mmf on the production of a flux that does not react back is analogous to the effect on the production of an airgap flux by the d-c excitation of a synchronous machine. Hence in analogy to Fig. 6.1*c* the equivalent circuit of Fig. 10.8 is established. As far as the *rotor* is concerned, Fig. 10.8 is similar to Fig. 10.7.

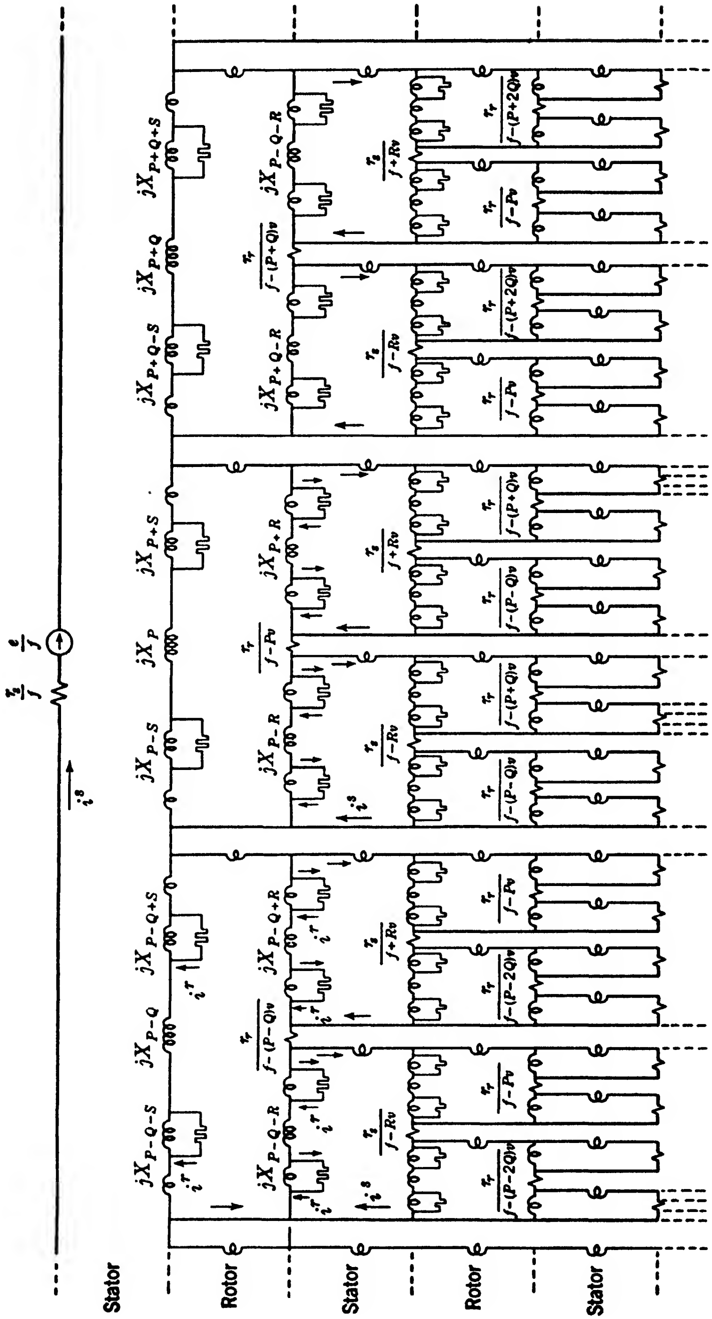



Fig. 10.7. Equivalent circuit of harmonics including *stator* slot openings.

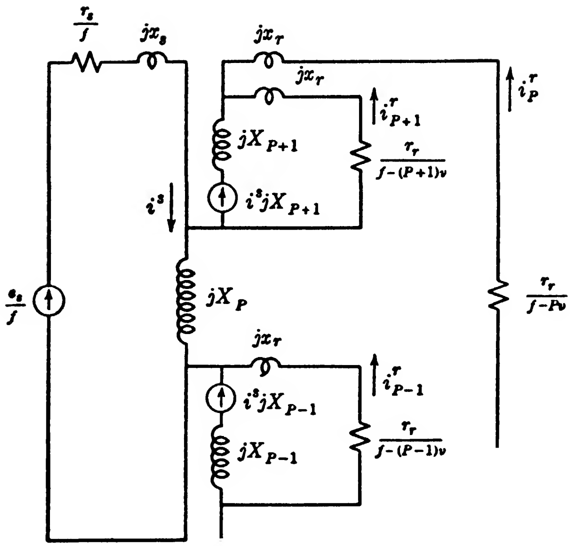



FIG. 10.8. Polyphase induction motor with eccentric rotor.

## HARMONICS DUE TO ROTOR-SLOT OPENINGS

The effect of the rotor-slot openings is analogous to that of the stator-slot openings. Each resultant mmf, such as $i_{P+R}$, produces with the *rotor* slot openings a series of fluxes with $P + R + kR$ pairs of poles. However, each of these fluxes has the same frequency with respect to the rotor as the mmf producing it.

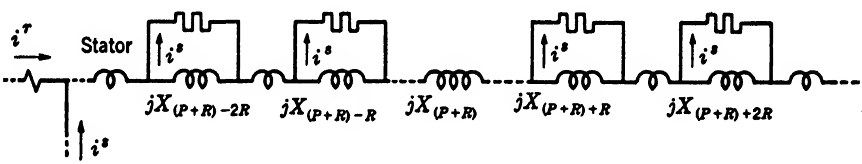



FIG. 10.9. Secondary chain of harmonics due to *rotor* slot openings.

These fluxes excite, beginning with the *stator*, a second set of secondary chain of harmonics, as shown in Fig. 10.9.

## BENT ROTOR SHAFTS

A special case of a rotor-slot permeance is a permanently bent rotor. An airgap permeance with *one* pair of poles appears which rotates with the rotor.

Though the resulting fluxes cut the stator winding as the first step in

the chain production, nevertheless in a winding with $P$ pairs of poles ($P$ is two or more) no voltages with $P \pm 1$ pairs of poles are generated. Hence these fluxes only act as leakage fluxes, without starting a chain of harmonics.

## HARMONICS DUE TO THE COMBINED PERMEANCES

Because of the *coexistence* of both stator- and rotor-slot openings,

$$P_s = \cos k_1Sx \qquad \text{and} \qquad P_r = \cos (k_2Rx - k_2Rvt)$$

a third set of secondary harmonic permeances appears, representing their product,

$$P_i = \cos [(k_1S + k_2R)x - k_2Rvt]$$

Again each resultant mmf wave, such as

$$\text{Mmf}_s = \cos [(P + R)x - (f + Rv)t]$$

produces with each of these permeances a flux wave

$$\phi_s = \cos [(P + R + k_1S + k_2R)x - (f + Rv - k_2Rv)t]$$

(All the above frequencies are with respect to the stator.)

However, *the frequency of these fluxes is not the same as the frequency of the mmf producing the fluxes*, either with respect to the stator or with

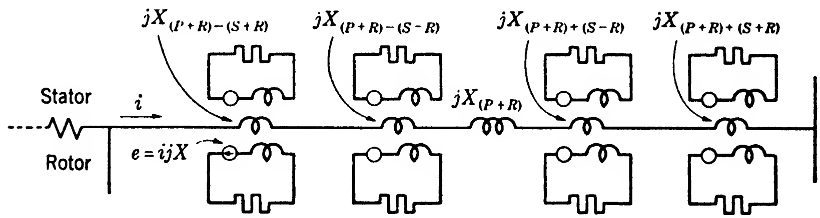



FIG. 10.10. Independent chain of harmonics due to *combined* slot openings.

respect to the rotor. (It should be recalled that the frequency of fluxes due to $P_s$ is the same as the frequency of the mmf producing the fluxes with respect to the stator, whereas the frequency of fluxes due to $P_r$ is the same as that of the mmf's with respect to the rotor.)

As these fluxes cut the stator and rotor, they start a new chain of harmonics (Fig. 10.10), which, however, *do not react upon the mmf's producing them.* The original network of Fig. 10.7 is not influenced by the chain of harmonics of Fig. 10.10, and the network and chain may be solved independently of each other.

The asymmetrical mutual inductance $jX$ is represented in the same manner as in Fig. 10.8, except that now asymmetricity exists with respect to the rotor mmf also.

Of course, the fluxes of Figs. 10.7 and 10.10 do interact in producing forces and noise. Also, it must be remembered that in subsynchronous crawlings the two independent sets of networks do interact. But the study of such an interaction is outside the scope of this book.

## THE NETWORK CALCULATION

In calculating the overall network, the lowest level of meshes in Fig. 10.7 should be solved first. Then its known impedance may be inserted into the parallel chain impedances along the line immediately above it, or two levels higher up. The same steps are repeated with each level.

Hence, in spite of the existence of secondary chains, it is sufficient to solve a network of the type in Fig. 10.5, in which each secondary chain is replaced by a parallel impedance. The parallel impedance is found as a by-product of the overall calculation and not as extra calculations.

## SPACE HARMONICS OF THE SHADED-POLE MOTOR

Since in a shaded-pole motor (Fig. 5.18$a$) the stator mmf wave is non-sinusoidal, the stator harmonic mmf's produce, in cooperation with the non-uniform airgap reactances $X_{md}$ and $X_{mq}$, a series of very large airgap fluxes with $P$, $3P$, $5P$, $7P$, etc., pairs of poles. These harmonic

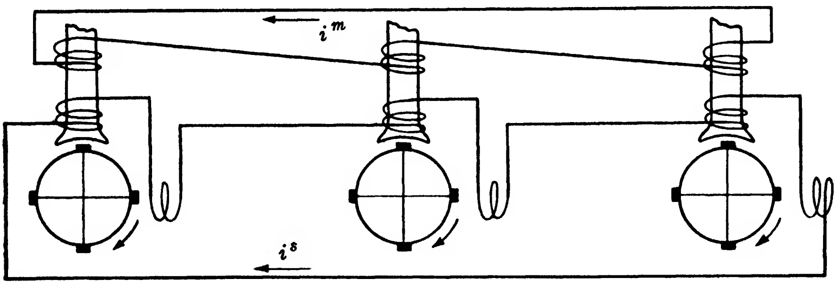



FIG. 10.11. Interconnection of three shaded-pole motors having $P$, $3P$, and $5P$ pairs of poles.

airgap fluxes all are independent of the angle of shift of the shaded coil, if the motor is replaced by an equivalent split-phase motor with windings at right angles.

Each of these fluxes cuts the rotor, producing in it a current-density and flux-density wave having the same number of pairs of poles as the

stator flux originating them. Hence each shaded-pole motor may be looked upon as consisting of several motors with a different number of pairs of poles, whose stator windings are connected in series (Fig. 10.11).

For each motor an equivalent circuit may be developed, using either the cross-field or the revolving-field reference frames. In these motors for each space harmonic the following properties apply:

1. The rotor resistance $r_r$ and leakage reactances $x_r$ differ by the factor $k_n$.

$$k_n = \frac{k_{(n)}{}^2}{k_{(1)}{}^2}$$

where $k_{(n)}$ is the $n$th harmonic pitch factor.

2. The mutual impedances $X_{md}$ and $X_{mq}$ differ by $k_n$ and also by $1/P^2$.

3. The stator leakage impedances (functions of the angle of shift $\alpha$) differ by unspecified amounts that depend on the particular design.

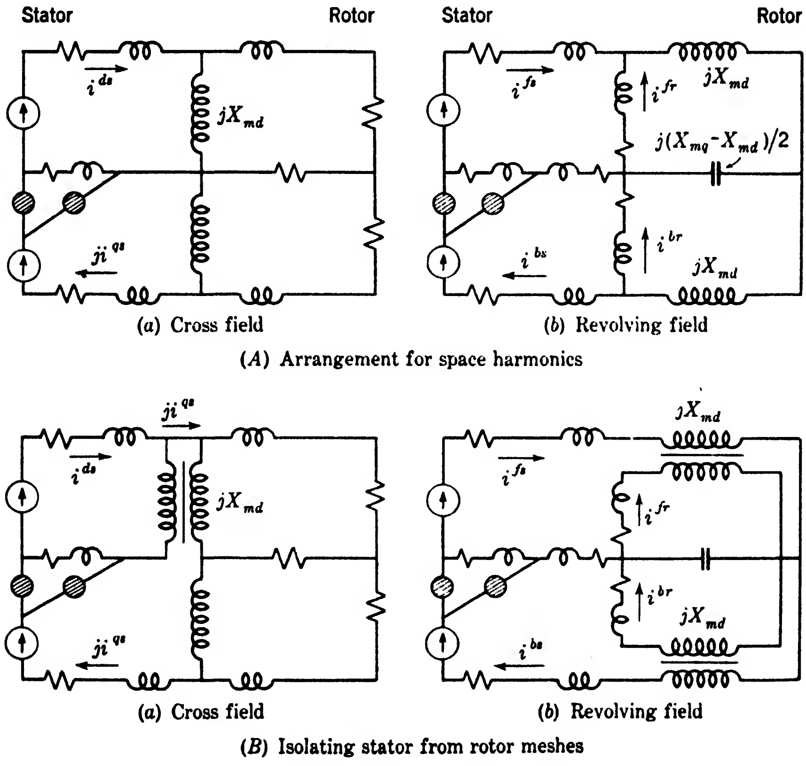



FIG. 10.12. Rearrangement of Figs. 5.19$b$ and 5.20

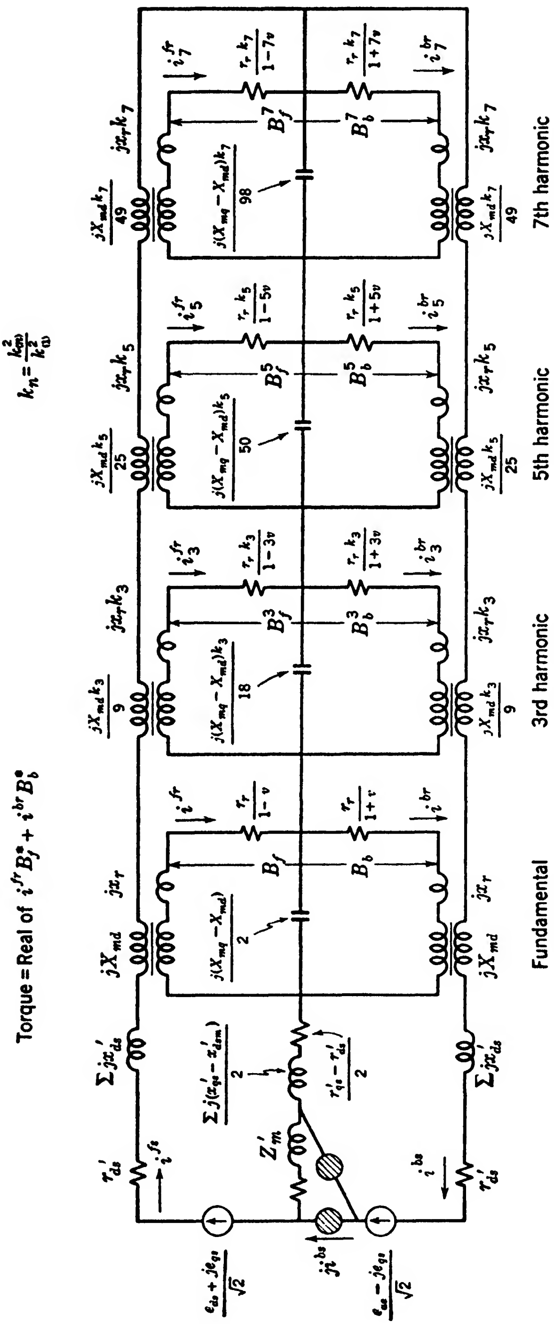



FIG. 10.13. Equivalent circuit of the shaded-pole motor including space harmonics (revolving-field theory).

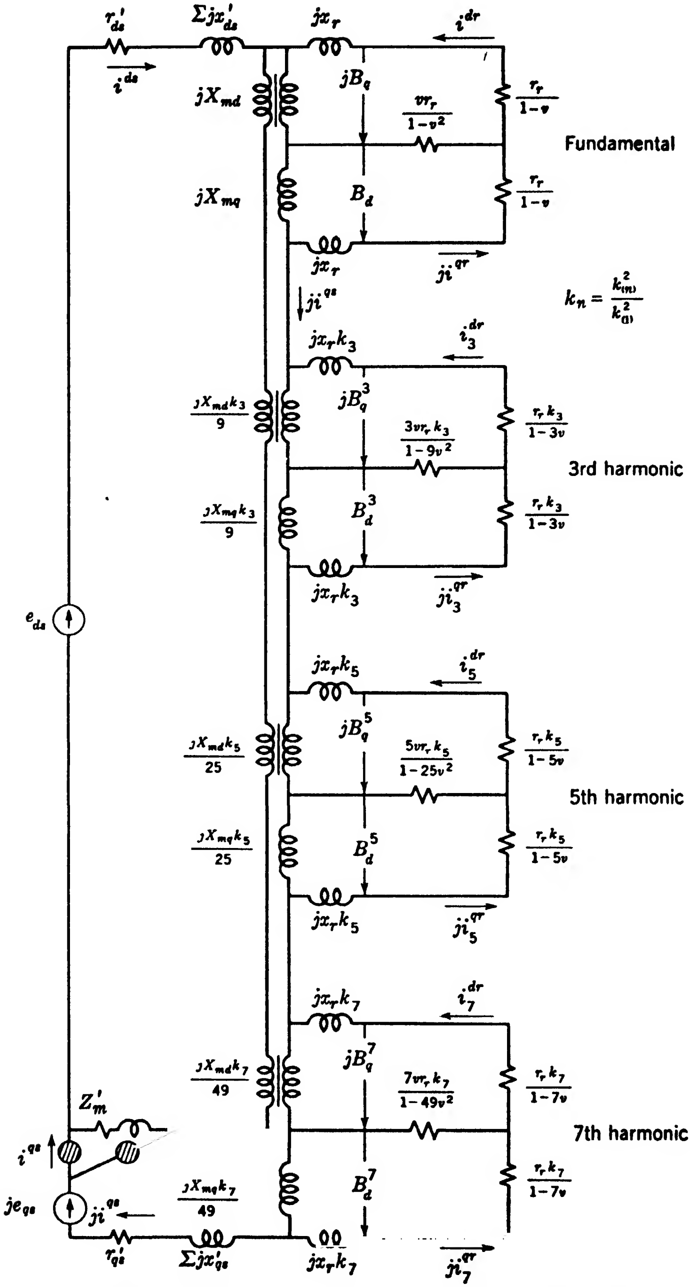



FIG. 10.14. Equivalent circuit of the shaded-pole motor including space harmonics (cross-field theory).

4. The rotor speeds $v$ differ by $P$.

In order to be able to connect the stator windings of the equivalent circuits in series, it is necessary to rearrange the circuits of Figs. 5.19 and 5.20 in two successive steps.

(*a*) In Fig. 10.12*A* the configuration of the coils of Fig. 5.20 is changed (Fig. 5.19 is left unchanged).

(*b*) In Fig. 10.12*B* some of the stator meshes of both networks are isolated from the rotor meshes by the use of ideal transformers.

Now the stator circuits are ready to be connected in series, as shown in Figs. 10.13 and 10.14. The resultant networks give the fundamental and harmonic rotor currents and torques as shown. It should be especially noted that *no quantity in the harmonic meshes depends on the angle of shift* $\alpha$. This angle influences only the stator resistances and leakage reactances.

## SOLUTION OF THE NETWORK

The network Fig. 10.14 of the cross-field theory is more adaptable for numerical calculation. In particular:

1. All quadrature-axis meshes of the rotor may be eliminated in succession by a simple mesh-star transformation.
2. In the remaining network each transformer may be replaced by a single mutual coil.
3. All direct-axis meshes of the rotor may be eliminated in succession simply by replacing two parallel impedances by a single impedance.

With the above simple steps the harmonic network is reduced to two meshes only.

The harmonics may also be calculated in an *approximate* manner, by solving first for the fundamental currents with the aid of the original simple networks of Figs. 5.19 and 5.20 and then using these currents to solve for the harmonic currents and fluxes.

## SPACE HARMONICS OF INDUCTION AND SYNCHRONOUS MACHINES

The fundamental equivalent circuits (Fig. 5.19 and 5.20) of the shaded-pole motor reduce to the equivalent circuits of all unbalanced or balanced induction and synchronous machines as special cases. A similar situation arises in the presence of space-harmonic waves in the stator. The space-harmonic equivalent circuits of Figs. 10.13 and 10.14 also reduce to those of all types of balanced or unbalanced induction and synchronous machines as special cases. Because of the rarity of their use no detailed analysis is undertaken here.

# 11 TIME HARMONICS

## ASYMMETRICAL STATOR AND ROTOR STRUCTURES

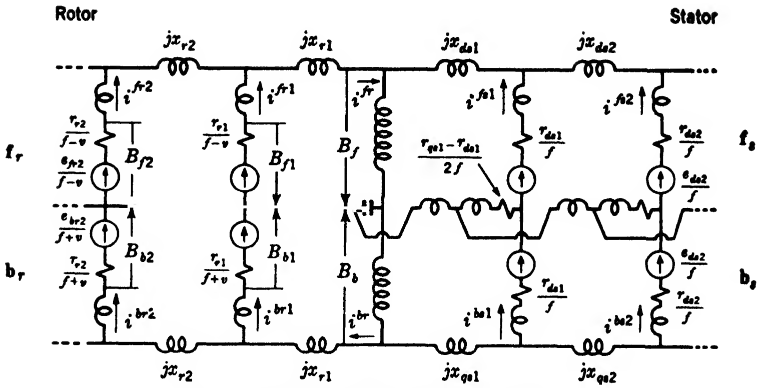
In all machines considered up to this point, it was assumed that either the stator structure or the rotor structure had balanced windings. That is, in all machines the resistances and leakage reactances were assumed to be the same along the **d** and **q** axes on either the entire stator structure (synchronous machines) or on the entire rotor structure (induction machines).

Similarly, when stationary networks were connected to the balanced portion of an unbalanced machine (loads to the slip rings of induction motors, or transmission lines to the armatures of synchronous machines), they also were assumed to be balanced two-phase networks. Moreover, when several machines running at different speeds were interconnected, all structures had to be balanced except one (the two single-phase selsyns out of synchronism, Fig. 9.17).

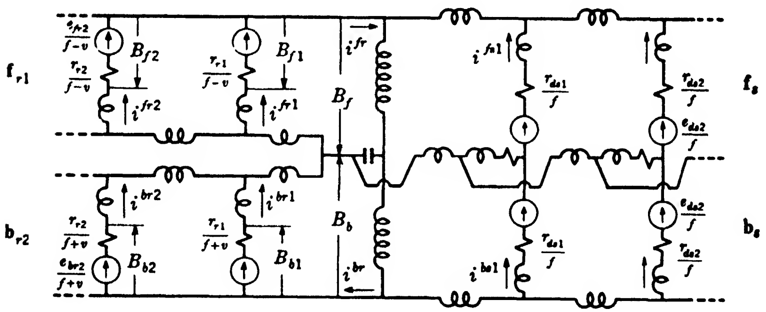
When both structures of a single machine are unbalanced, *an infinite number of time harmonics appear in the whole system*, producing non-sinusoidal resultant currents in time, even though the impressed voltages are sinusoidal in time. The most familiar example of an industrial machine, which is non-sinusoidal even during steady state, is the single-phase alternator. Other examples are slip-ring induction motors operated single-phase on both stator and rotor, unbalanced faults on synchronous machines, and instantaneous short circuits of synchronous and induction machines.

## ROTOR REFERENCE FRAMES ATTACHED TO THE STATOR

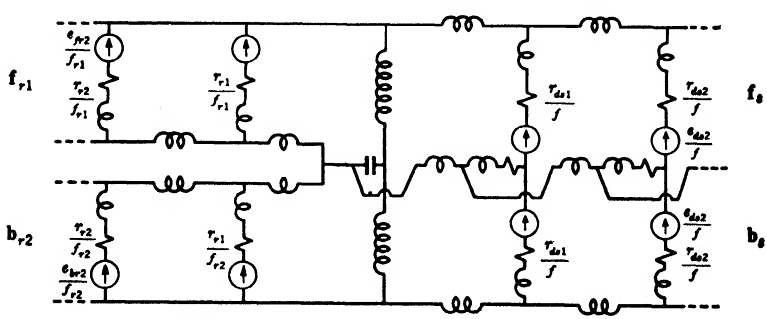
Let the equivalent circuit of the primitive induction machine be reproduced on Fig. 11.1*a*. Attention is called to the fact that because the stator windings are unbalanced ($r_{ds1} \neq r_{qs1}$) an **f** current induces a **b** voltage by means of the unbalanced value of the impedances $(r_{qs1} - r_{ds1})/2$, etc., that appear in the common branch. If the stator windings were balanced, the common branch in the stator would be as bare of impedances as the rotor common branch.




(a) Rotor reference frames attached to stator




(b) Rotor reference frames attached to rotor




(c) Same as Fig. b but with three base frequencies $f$, $f_{r1}$, $f_{r2}$

FIG. 11.1. Two points of view of the primitive induction machine (revolving-field network).

It should also be noted that in the stator both **f** and **b** meshes have the same absolute frequency $f$, whereas in the rotor the **f** and **b** meshes have different absolute frequencies. In particular, an **f** mesh possesses an $f - v$ absolute frequency only, and its corresponding **b** mesh (with $f - v$ absolute frequency) is missing, because the rotor structure is smooth and has balanced windings. Similarly, in the rotor only a **b** mesh is associated with the absolute frequency $f + v$. Its corresponding **f** mesh is missing for the same reasons.

## DEVICES TO INTRODUCE ROTATING REFERENCE FRAMES

In interconnecting several machines, rotating reference frames have been introduced by the device of changing the base frequency $f$ (the frequency along the reference frame assumed) from the stator frequency $f_s$ to the rotor frequencies $f_{r1} = f - v$ and $f_{r2} = f + v$, *leaving the values of the absolute frequencies unchanged.* A variable phase shifter was also used to emphasize the change in the base frequencies, but its presence was not mandatory.

Now a new device will be introduced to indicate the introduction of rotating reference frames. Instead of emphasizing the appearance of new base frequencies, *the rotor* **f** *and* **b** *meshes will be rearranged* in such a manner that a set of **f** and **b** meshes, denoting a set of rotor physical reference frames, should have *identical* absolute frequencies.

## ROTOR REFERENCE FRAMES ATTACHED TO THE ROTOR

Let it be assumed that the two physical reference frames associated with the rotor are not stationary in space but are rigidly attached to the rotor structure. When the rotor rotates, the two rectangular axes, to be denoted by $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, also rotate. In two-phase wound rotors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ represent the actual rotor windings.

(In his previous writings the author denoted these $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ physical reference frames by **a** and **b**, since the corresponding three-phase axes are denoted by **a**, **b**, **c**. However, the letter *b* already has been reserved in this book for the backward axes and the letter *a* for the armature. Since the use of $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ has been customary in synchronous-machine literature for these same physical frames, that usage will be adopted here also. Nevertheless, it is objectionable to use both Roman and Greek letters to denote physical reference frames of the same nature.)

In connection with this new set of rotor physical reference frames, $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$, it is possible to introduce a set of hypothetical sequence axes $\mathbf{f}_r$ and $\mathbf{b}_r$, which are *not identical* with the previous sequence axes **f** and **b** associated with the stationary physical axes $\mathbf{d}_r$ and $\mathbf{q}_r$.

## FREQUENCIES ALONG THE NEW FRAME

Along the physical reference frame **d** and **q** (attached to the stator) the base frequency of rotor currents is $f$, but along the physical frame $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (attached to the rotor)—and also in the actual rotor windings —there exist *two different base frequencies of rotor currents,* $f_{r1} = f - v$ and $f_{r2} = f + v$. Hence *two sets of physical axes* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *may be attached to the rotor, one set* $(\boldsymbol{\alpha}_1\boldsymbol{\beta}_1)$ *for the* $f_{r1}$ *frequencies, the other set* $(\boldsymbol{\alpha}_2\boldsymbol{\beta}_2)$ *for the* $f_{r2}$ *frequencies.* Now on both stator and rotor the base frequencies are identical with the absolute frequencies.

(The appearance of one frequency on the stator and two frequencies on the rotor is accidental. Had the $f$ frequency voltage been impressed in the *actual* rotor windings [along $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$], instead of the actual stator windings, the role of frequencies would have been interchanged between the stator and rotor. This latter case will be considered presently.)

Associated with the two sets of rotor physical axes are two sets of sequence axes. But, *of the two sequence axes* $(\mathbf{f}_{r1}, \mathbf{b}_{r1})$ *associated with the* $f_{r1} = f - v$ *frequency currents, only one exists,* the forward mesh $\mathbf{f}_{r1}$. Also, of the two $f_{r2} = f + v$ frequency sequence axes $(\mathbf{f}_{r2}, \mathbf{b}_{r2})$ only the backward mesh, $\mathbf{b}_{r2}$, exists. The two sequence axes are missing, because the two rotor windings $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are balanced windings, and a forward current of any frequency does not generate a backward current of the same frequency, and vice versa.

## EQUIVALENT CIRCUIT ALONG THE NEW FRAME

To summarize, the old point of view attributes the $\mathbf{f}_r$ and $\mathbf{b}_r$ rotor meshes to *one* set of stationary rotor physical axes, $\mathbf{d}_r$ and $\mathbf{q}_r$, rigidly attached to the stator structure. On the other hand, the new point of view attributes the $\mathbf{f}_r$ mesh to a set of physical axes rigidly attached to the rotor, and having an $f - v$ frequency *forward* current only, with its backward component missing. Similarly, the $\mathbf{b}_r$ mesh is attributed to a second set of physical axes rigidly attached to the rotor and having an $f + v$ frequency *backward* current only, with its forward component missing.

To show the new point of view in the equivalent circuit, let the rotor meshes in Fig. 11.1*a* be merely flipped over by interchanging the slot-leakage reactances with the bare common branch, as shown in Fig. 11.1*b*. *Now there are two bare common branches in the rotor,* indicating that the $f - v$ frequency $\mathbf{f}_{r1}$ mesh has no corresponding $\mathbf{b}_{r1}$ mesh and that the $f + v$ frequency $\mathbf{b}_{r2}$ mesh has no corresponding $\mathbf{f}_{r2}$ mesh in the presence of a balanced rotor layer of winding.

The change of point of view (reference frame) merely rearranges the network configuration without changing the values of the impedances

or the absolute frequencies. This rearrangement will facilitate the introduction of time harmonics.

## THE BASE FREQUENCIES

In the presence of rotating rotor frames there are three base frequencies, namely, $f_s$, $f_{r1}$, and $f_{r2}$. The relations between them are $f_{r1} = f_s - v$ and $f_{r2} = f_s + v$. In the equivalent circuit of Fig. 11.1*b* there are also three different absolute frequencies (the denominators of $r$ and $e$). These absolute frequencies may be expressed either in terms of the stator base frequency $f_s = f$, as it is done in Fig. 11.1*b*, or in terms of any one of the other base frequencies, or in terms of all three base frequencies, as shown in Fig. 11.1*c*. In the rest of the book the absolute frequencies will be expressed always in terms of only one of the base frequencies, usually that of the stator impressed-voltage frequency.

It should be noted that, if an $f$ frequency voltage is impressed in the actual *rotor* winding, instead of in the stator, the distribution of absolute frequencies in the equivalent circuit is different from the distribution that exists when the same $f$ frequency voltage is impressed in the actual *stator* windings. Hence a different equivalent circuit will have to be used, depending on whether the $f$ frequency voltages are impressed in the actual stator windings (along $\mathbf{d}$, $\mathbf{q}$) or in the actual rotor windings (along $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$).

When all absolute frequencies are expressed in terms of only *one* base frequency $f$ (and several velocities), the equivalent circuit represents also the *transient* behavior of the machine or group of machines simply by replacing the single $f$ by $-jp$, as shown in detail in Appendix 1.

## UNBALANCE IN THE ROTOR WINDINGS ONLY

Let it be assumed now temporarily that in an induction motor the stator windings are balanced and the stator structure is smooth. In their place let the rotor windings $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ be unbalanced and let the rotor structure have saliency. In that case $r_{\beta r1}$ is different from $r_{\alpha r1}$; also $x_{\beta r1}$ differs from $x_{\alpha r1}$; etc. In analogy to the unbalanced stator windings, the differences $(r_{\beta r1} - r_{\alpha r1})/2$, $(x_{\beta r1} - x_{\alpha r1})/2$, etc., appear in the common branches of Fig. 11.2 that previously were bare. Hence the missing $\mathbf{b}_{r1}$ and $\mathbf{f}_{r2}$ rotor meshes reappear on the rotor, as shown in Fig. 11.2.

However, these missing rotor meshes are coupled to the stator through the airgap reactances $X_{md}$; hence, in the stator also, two new meshes appear. Just as previously the two $f$ frequency stator meshes induced in the rotor $f - v$ and $f + v$ frequencies, now the two $f - v$ frequency rotor meshes must induce in the stator $(f - v) - v = f - 2v$ and

$(f - v) + v = f$ frequencies. Similarly, the two $f + v$ frequency rotor currents must induce in the stator $(f + v) - v = f$ and $(f + v) + v = f + 2v$ frequencies, as shown in Fig. 11.2.

*As the stator is balanced and is smooth, the production of harmonics stops at this stage* and the equivalent circuit of Fig. 11.2 is a self-contained system, representing a balanced, smooth stator with **d**, **q** axes and a

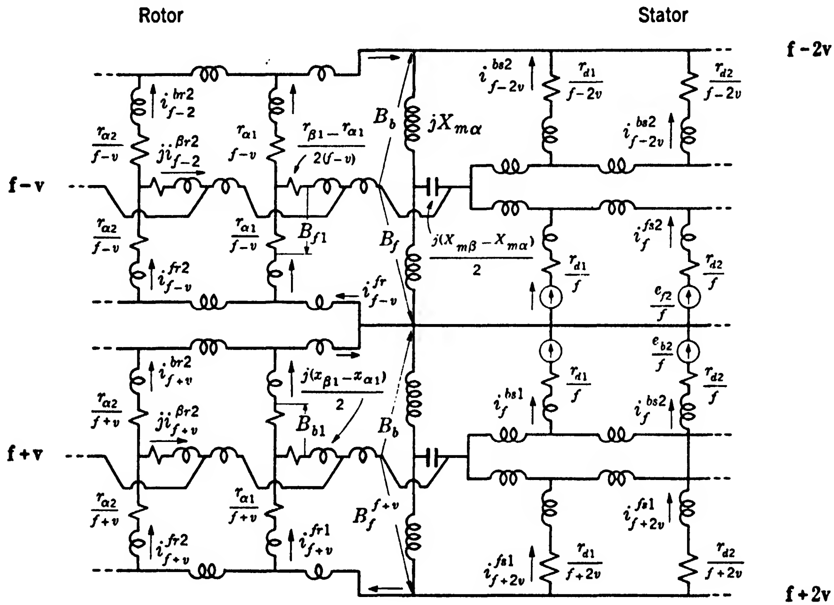



FIG. 11.2. New primitive induction machine; stator balanced, rotor unbalanced (revolving-field network).

salient-pole unbalanced rotor with **α**, **β** axes. Although the circuit contains sixteen meshes, *it splits up into two independent eight-mesh circuits* at its center horizontal line. In one circuit only a positive-sequence stator voltage $e_{sf}$, is impressed, in the other only a negative-sequence voltage $e_{sb}$, both of fundamental frequency $f$.

Other stator and rotor voltages may be simultaneously impressed only if their absolute frequencies are the same as the absolute frequencies of the corresponding meshes in the equivalent circuit.

## POLYPHASE INDUCTION MOTOR WITH UNBALANCED LOAD ON ITS ROTOR

A special case of the primitive induction machine is a polyphase induction motor (a motor with balanced stator windings) whose slip rings are connected to an *unbalanced* load (that may include also a

capacitor). The equivalent circuit is shown in Fig. 11.3. It consists of two independent networks, one containing the forward, the other containing the backward impressed stator voltages. The fundamental frequency $f$ excitation exists only on the stator. On the slip rings a forward $f - v$ frequency voltage may be impressed.

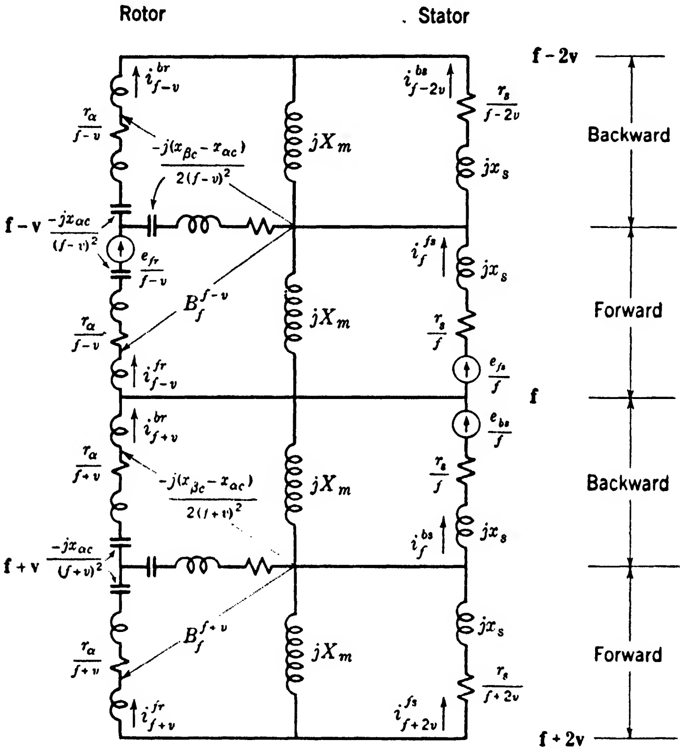



Fig. 11.3. Polyphase induction motor with unbalanced rotor load.

It should be especially noted that the unbalanced portions of the load $(r_\beta - r_\alpha)/2$ are divided by the same absolute frequency $f - v$ (or $f + v$), as the balanced portion $r_\alpha$. That is, *on both sides of an unbalanced portion the same absolute frequencies must exist, in order that a forward current should generate a backward current of the same frequency.*

## POLYPHASE INDUCTION MOTOR WITH SINGLE-PHASE ROTOR

Although the equivalent circuits of Figs. 11.2 and 11.3 contain only sequence currents, some of the *physical* currents also can be defined on them. In particular, in the common rotor branches flow each of the two actual rotor currents of the β winding $ji^{\beta r}$. Hence *when one of the*

*actual rotor windings* β *is missing, the common rotor branch becomes open-circuited, as shown in Fig. 11.4 for a polyphase induction motor with a*

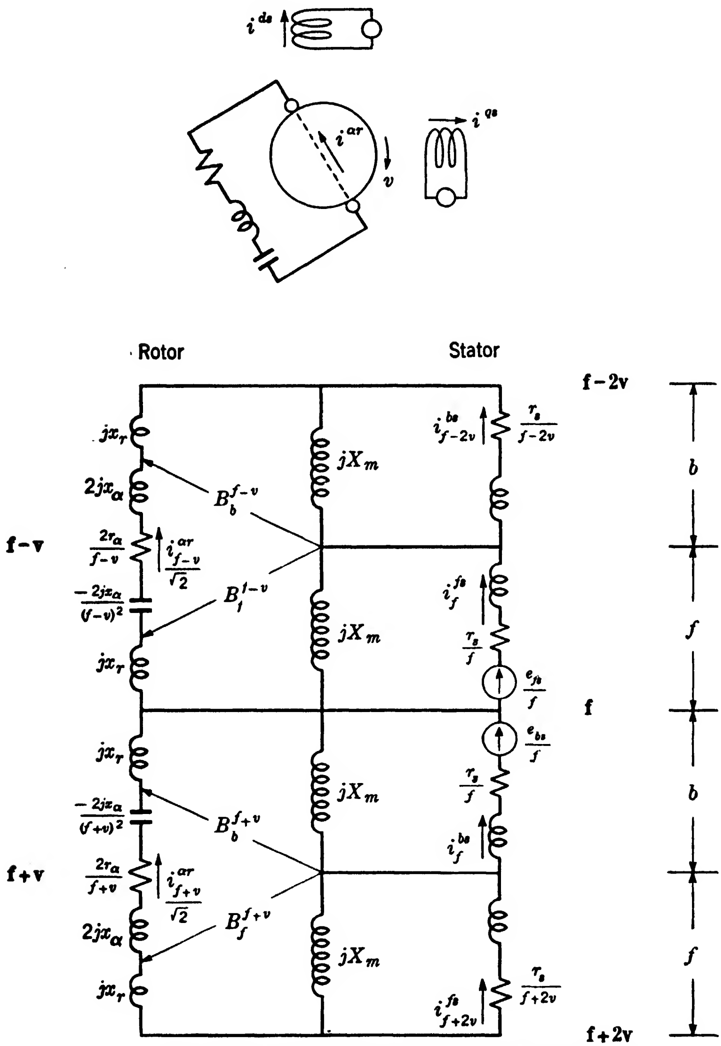



FIG. 11.4. Polyphase induction motor with single-phase rotor.

*single-phase rotor. With only a forward stator voltage impressed, the lower half of the network disappears.*

The single-phase-rotor flux density along axis α—needed in the torque calculations—is found as the *sum* of $B_f$ and $B_b$, since $B_\alpha = (B_f + B_b)/\sqrt{2}$. The open-circuit flux leakage $jB_\beta$ (and hence the

*open-circuit voltage* along axis β) is found to be the difference between $B_f$ and $B_b$ by means of $B_\beta = -j(B_f - B_b)/\sqrt{2}$.

## SYNCHRONOUS MACHINE WITH BALANCED ARMATURE

For a synchronous machine with balanced armature an analogous circuit applies (Fig. 11.5). Now *the armature reference axes are rigidly connected to the armature.* Also, excitation exists only on the armature.

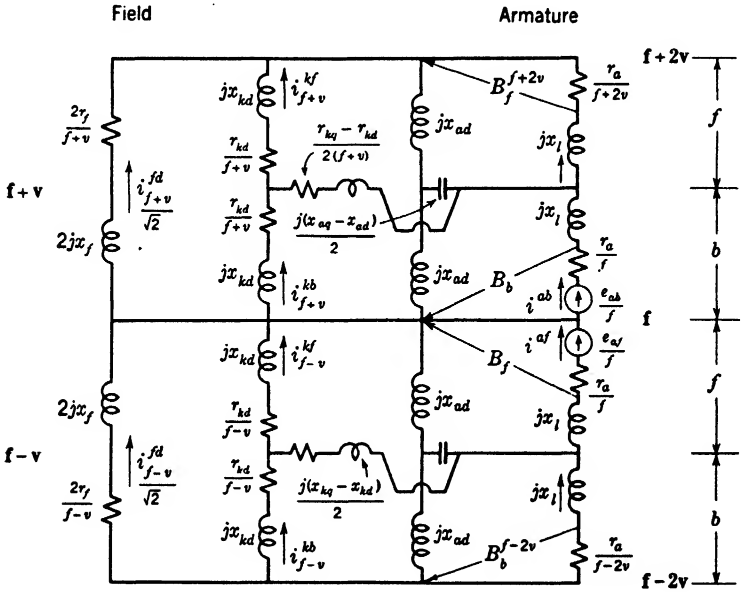



FIG. 11.5. Synchronous machine. (Armature reference frame connected to the armature. Excitation on armature only.)

This network with two independent circuits is identical with the two circuits of Figs. 6.7*Aa* and *Bc*. The difference lies only in the value of the impressed frequency. Here the frequency of the positive sequence voltage $e_{sf}$ is $f$, whereas in Fig. 6.7*Aa* it was $f - v$ (or $1 - v$). Here the frequency of the negative-sequence voltage $e_{sb}$ is still $f$, whereas in Fig. 6.7*Bc* it was $f + v$ (or $1 + v$).

Note the change in sign of $v$ in Figs. 11.3 and 11.5.

## TWO POLYPHASE INDUCTION MOTORS (SELSYNS) WITH UNBALANCED LOADS RUNNING AT DIFFERENT SPEEDS

Let two induction motors with balanced stator windings run at different speeds and let the load between them be unbalanced (Figs. 9.15

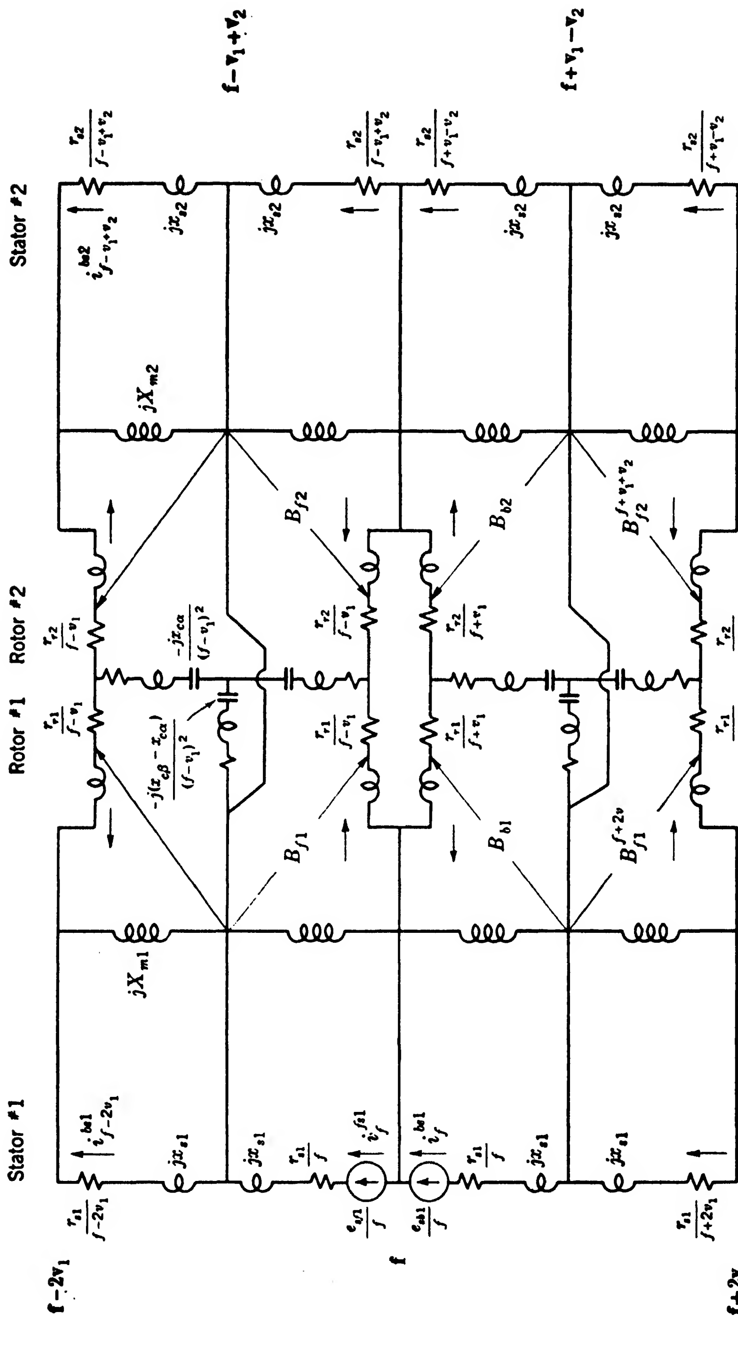



FIG. 11.6. Two polyphase induction motors with unbalanced loads running at different speeds.

and 11.6). In the independent circuit with the forward emf impressed, $f_2$ is found by equating the two interconnected rotor frequencies $f_1 - v_1$ and $f_2 - v_2$. The equation gives $f_2 = f_1 - v_1 + v_2$. In the independent circuit with the backward emf impressed $f_1 = f_2 + v_1 - v_2$.

A separate network is needed for each machine excitation. If $v_2 = v_1 = v$ then both excitations may be applied on the same network.

## ONE STRUCTURE SMOOTH, THE OTHER UNBALANCED

All cases considered in this chapter hitherto could have been analyzed also without the reinterpretation of the rotor reference frames simply by using the standard equivalent circuits of Chapter 3. By impressing in the standard circuits only a forward or only a backward voltage with a base frequency of $f - v$ or $f + v$ (instead of $f$), the equivalent circuits presented thus far in this chapter could have been found also.

However, the reinterpretation of the speed of the rotor physical reference frame has introduced the concept of *interrelation of two primitive networks* differing from each other only in absolute frequencies. The foundation has been thereby laid to interrelate in a logical manner *several* primitive networks and thereby to assume a machine with unbalance in both its stator and rotor.

## UNBALANCED STATOR AND ROTOR STRUCTURES

Returning to Fig. 11.2, with the eight meshes, let it be assumed that the stator windings are also unbalanced. However, *the saliency may be retained either on the stator only or on the rotor only, but not on both structures*, as a double saliency would introduce complications into the equivalent circuit on denoting the airgap reactances along the vertical branch $X_{md}$ or $X_{m\alpha}$. (Although there is room in the common branches for $X_{mq} - X_{md}$ and $X_{m\beta} - X_{m\alpha}$, no room is left for both $X_{md}$ and $X_{m\alpha}$ in the *vertical* branches, unless either the saliency is restricted to one member, or the equivalent circuit is generalized still further.)

When the stator windings are unbalanced, the smooth common branch on the stator equivalent circuit contains difference impedances and the missing stator $\mathbf{f}_{s2}$ mesh with $f - 2v$ frequency reappears. At the same time the missing stator $\mathbf{b}_{s2}$ mesh with $f + 2$ frequencies also reappears, as shown in Fig. 11.7. Now, that both **f** and **b** meshes of $(f - 2v)$ frequency exist on the stator, they induce in the rotor two currents, one with $(f - 2v) + v = f - v$ frequency and another with $(f - 2v) - v = f - 3v$ frequency.

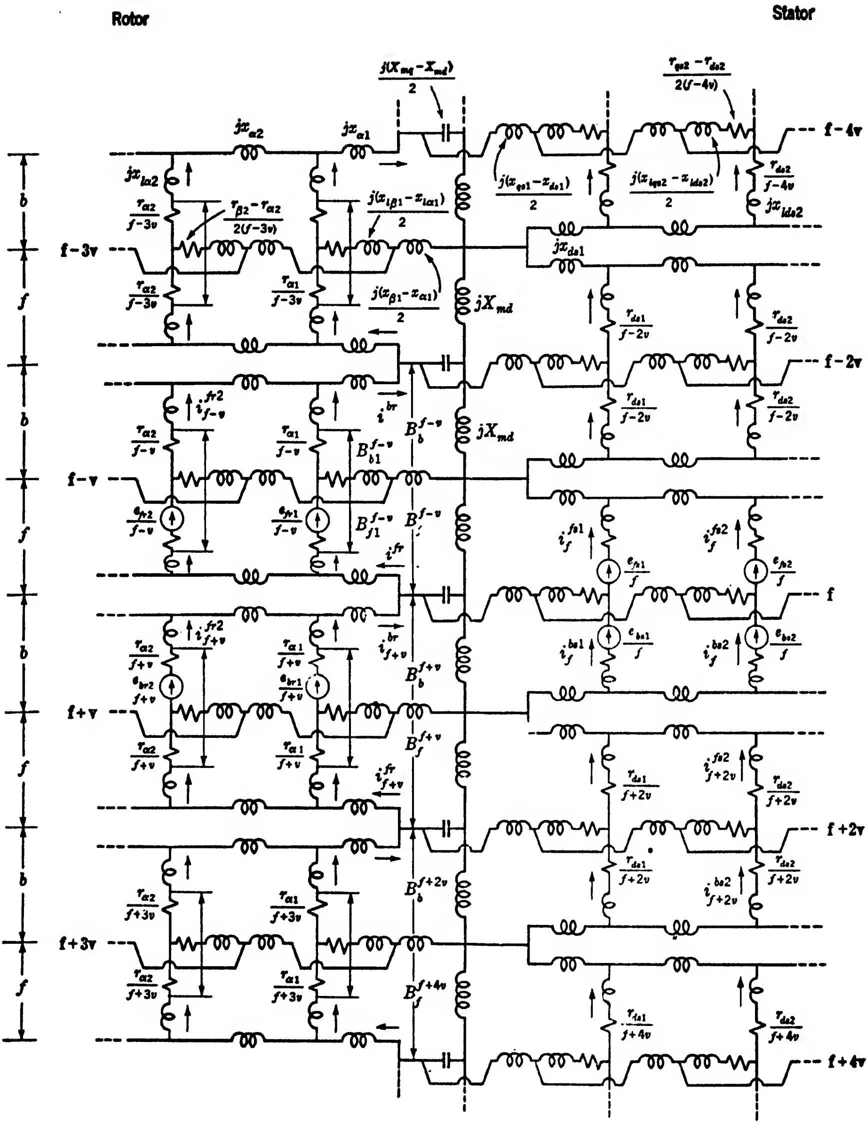



FIG. 11.7. The primitive induction machine with unbalanced stator and rotor (revolving-field network).

## THE UNBALANCED PRIMITIVE INDUCTION MACHINE.* REVOLVING-FIELD NETWORK

If the addition of stator and rotor meshes is continued by increasing and decreasing their absolute frequency with $v$, the equivalent circuit of the unbalanced primitive induction machine of Fig. 11.7 is finally established. For the unbalanced *synchronous* machine $v$ and $B$ change signs. (Again it should be noted that the saliency may occur either on the stator [as in Fig. 11.7] or on the rotor [as in Fig. 11.2], but not on both.)

In the equivalent circuit of Fig. 11.7 it is assumed that a voltage of $f$ frequency is impressed on the *stator* only. However, it may well be that the voltage of the $f$ frequency is impressed on the *rotor* only. Then either all $f$'s in Fig. 11.7 are replaced by $f - v$ (or $f + v$), or the roles of stator and rotor are simply interchanged. In general, *the rotor impressed voltages require a separate equivalent circuit from those of the stator impressed voltages*, unless the frequencies of the two sets of voltages match the absolute frequencies dictated by the equivalent circuit. For instance, in a doubly fed induction motor, whose stator is excited by an $f$ frequency forward voltage and its slip rings on the rotor are excited by a slip frequency, $f - v$, forward voltage, both voltages may be impressed simultaneously on the same network.

*In the primitive equivalent circuits of Fig. 11.7 the distinction between stator and rotor have disappeared, as both structures have the same form of common branches containing the difference between unequal impedances.* The stator and rotor differ only in absolute frequencies, owing to the location of the impressed voltage. Of course, the absolute frequencies may be expressed, if it is so desired, in terms of a base frequency other than that of the stator $f_s = f$.

It should be noted that the value of all reactances is identical in all harmonic meshes and that only the resistances vary from one harmonic to the other.

Whatever special cases were established from the *balanced primitive network* of Fig. 3.1 while the various types of induction, synchronous, and commutator machines and their interconnection were considered, the same special cases apply to the more general *unbalanced primitive networks* of Fig. 11.7. Of course, in practice the need for consideration of harmonic currents is less frequent and less urgent.

*No cross-field equivalent circuit will be developed in this book* for machines having unbalanced stator and rotors.

* Gabriel Kron, "Equivalent Circuit of the Primitive Machine with Asymmetrical Stator and Rotor," *Transactions of the AIEE*, Vol. 66, pp. 17–23, 1947.

## THE HARMONIC TORQUE CALCULATIONS

The various harmonic torques are established by the formulas of Chapter 4 without any change. Considering the case of Fig. 11.7, where *only the stator is excited*, the resultant rotor currents are

$$\begin{aligned} i^f &= i_{f-3v}{}^f + i_{f-v}{}^f + i_{f+v}{}^f + i_{f+3v}{}^f + \cdots \\ i^b &= i_{f-3v}{}^b + i_{f-v}{}^b + i_{f+v}{}^b + i_{f+3v}{}^b + \cdots \end{aligned} \qquad 11.1$$

Similar expressions apply to the resultant rotor flux densities.

The sum-frequency torque is, by Eq. 4.7,

$$T_+ = i^f B_b + i^b B_f$$

Leaving out the indices, two such products have to be formed as

$$\begin{aligned} T_+ = (\cdots &+ i_{f-3v} + i_{f-v} + i_{f+v} + i_{f+3v} + \cdots) \\ &(\cdots + B_{f-3v} + B_{f-v} + B_{f+v} + B_{f+3v} + \cdots) \end{aligned} \qquad 11.2$$

Table 11.1 groups the products according to the *sum* of the subscripts, representing the torque frequencies.

TABLE 11.1 SUM-FREQUENCY TORQUES

| $T_+ =$ | $B_{f-3v}$ | $B_{f-v}$ | $B_{f+v}$ | $B_{f+3v}$ |
|---|---|---|---|---|
| $i_{f-3v}$ | $2f - 6v$ | $2f - 4v$ | $2f - 2v$ | $2f$ |
| $i_{f-v}$ | $2f - 4v$ | $2f - 2v$ | $2f$ | $2f + 2v$ |
| $i_{f+v}$ | $2f - 2v$ | $2f$ | $2f + 2v$ | $2f + 4v$ |
| $i_{f+3v}$ | $2f$ | $2f + 2v$ | $2f + 4v$ | $2f + 6v$ |

The table shows what harmonic currents and fluxes should be multiplied together to get the harmonic torques of the same frequencies. For instance, the $2f + 2v$ frequency torques are

$$T_{2f+2v} = i_{f+3v}B_{f-v} + i_{f+v}B_{f+v} + i_{f-v}B_{f+3v} + \cdots \qquad 11.3$$

Each expression is a complex number.

The difference-frequency torque is found by Eq. 4.8.

$$T_- = i^{f*}B_f + i^{b*}B_b \qquad 11.4$$

Leaving out the indices, two such products are formed as

$$T_- = (\cdots + i^*_{f-3v} + i^*_{f-v} + i^*_{f+v} + i^*_{f+3v} + \cdots)$$
$$(\cdots + B_{f-3v} + B_{f-v} + B_{f+v} + B_{f+3v} + \cdots) \quad 11.5$$

Table 11.2 groups the products according to the *difference* of the subscripts, representing the torque frequencies.

TABLE 11.2 DIFFERENCE-FREQUENCY TORQUES

| $T_- =$ | $B_{f-3v}$ | $B_{f-v}$ | $B_{f+v}$ | $B_{f+3v}$ |
|---|---|---|---|---|
| $i^*_{f-3v}$ | 0 | $2v$ | $4v$ | $6v$ |
| $i^*_{f-v}$ | $-2v$ | 0 | $2v$ | $4v$ |
| $i^*_{f+v}$ | $-4v$ | $-2v$ | 0 | $2v$ |
| $i^*_{f+3v}$ | $-6v$ | $-4v$ | $-2v$ | 0 |

By allowing $i$ and $B$ to assume the superscripts $f$ or $b$, as given by Eqs. 4.7 and 4.8, all the harmonic torques may easily be determined as complex numbers, $T' + jT''$. The peak value of the harmonic torque is given by Eq. 4.6:

$$T_{peak} = \sqrt{\Sigma(T')^2 + \Sigma(T'')^2} \quad 11.6$$

Let it next be assumed that the $f$ frequency voltage has been impressed on the *rotor* slip rings only. Then *the roles of the stator and rotor are simply interchanged,* and in the rotor even frequency ($2nv$) currents and voltages appear. For the frequency of torques similar tables may be established. In particular,

TABLE 11.3 SUM-FREQUENCY TORQUES

| $T_+ =$ | $B_{f-4v}$ | $B_{f-2v}$ | $B_f$ | $B_{f+2v}$ |
|---|---|---|---|---|
| $i_{f-2v}$ | $2f - 6v$ | $2f - 4v$ | $2f - 2v$ | $2f$ |
| $i_f$ | $2f - 4v$ | $2f - 2v$ | $2f$ | $2f + 2v$ |
| $i_{f+2v}$ | $2f - 2v$ | $2f$ | $2f + 2v$ | $2f + 4v$ |
| $i_{f+4v}$ | $2f$ | $2f + 2v$ | $2f + 4v$ | $2f + 6v$ |

TABLE 11.4 DIFFERENCE-FREQUENCY TORQUES

$T_- =$

| | $B_{f-4v}$ | $B_{f-2v}$ | $B_f$ | $B_{f+2v}$ |
|---|---|---|---|---|
| $i^*_{f-4v}$ | 0 | $2v$ | $4v$ | $6v$ |
| $i^*_{f-2v}$ | $-2v$ | 0 | $2v$ | $4v$ |
| $i^*_f$ | $-4v$ | $-2v$ | 0 | $2v$ |
| $i^*_{f+2v}$ | $-6v$ | $-4v$ | $-2v$ | 0 |

*The frequencies of torques are the same in the presence of a stator or a rotor excitation.* However, the combination of the harmonic currents and fluxes is different for the two cases.

It should be recalled that, if the torque frequency comes out a negative number (like $-2v$ for $i^*_{f-2v}B_{f-4v}$), then the frequency must be made positive by taking the conjugate of the torque (namely, $i_{f-2v}B^*_{f-4v}$).

## THE PHYSICAL REFERENCE FRAMES OF THE TIME HARMONICS

In the *balanced* primitive induction machine the physical reference axes **d** and **q** are stationary in space on both stator and rotor. Along these physical axes all stator and rotor quantities have fundamental $f$ frequencies. Associated with these stationary physical axes two hypothetical $\mathbf{f}_0$ and $\mathbf{b}_0$ axes ("sequence" or "spin" axes) are introduced, along which the equivalent circuit quantities are expressed.

The question now arises: What are the *physical* axes along which the harmonic reference axes of the revolving-field network of Fig. 11.7 are expressed?

Considering the two sets of *stator* meshes bordering the two sets of $f$ frequency meshes, the lower **f** mesh has $f + 2v$ frequencies, the upper **b** mesh has $f - 2v$ frequencies. Hence, *the* **d** *and* **q** *physical reference frame, with which they are associated, must be rotating with 2v velocity with respect to the stationary stator.* (That is, the *relative* velocities between the stator conductors and the reference-axes must be $2v$.)

The next two bordering meshes have the same two frequencies, with **f** and **b** interchanged in frequencies. Hence they are associated with two physical axes rotating with $-2v$ velocities. Each two of the bordering stator meshes correspond in succession to axes rotating with $4v$ and $-4v$ velocities with respect to the stator, and so on, through all *even* $v$ velocities.

With respect to these rotating frames all harmonics have fundamental $f$ frequencies.

## THE "HOLONOMIC" REFERENCE FRAME

In the derivation of Fig. 11.7 it was assumed that, on the rotor, the fundamental set of frames was rigidly connected to the rotor structure.

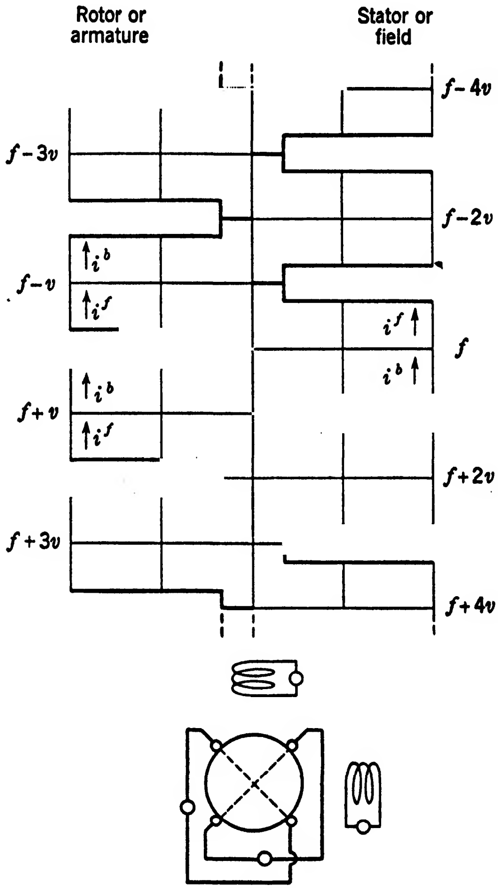



FIG. 11.8. Holonomic reference frames.

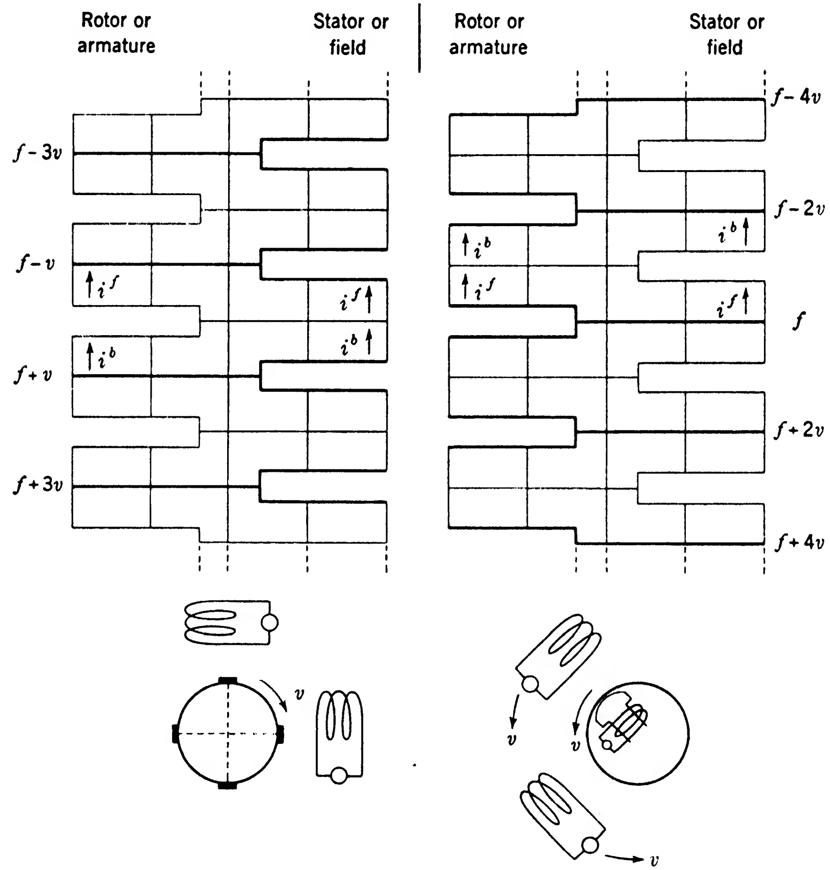
Hence all other rotor frequencies differ again by $2v$ only. Their reference frames rotate with $2nv$ velocities with respect to the rotor, Fig. 11.8. This fundamental system of reference frames (the stator frame rigidly connected to stator, and the rotor frame rigidly connected to the rotor) was originally introduced by Maxwell.

In the dynamical treatment of the theory of rotating machinery with the aid of the dynamical equations of Lagrange (see Epilogue), this same system of reference frame is called the "holonomic" reference frame.

## THE "NON-HOLONOMIC" REFERENCE FRAME

Since the basic network of Fig. 11.7 contains only absolute frequencies, there is nothing to prevent the introduction of two other types of reference frames:

1. It may be assumed that the fundamental *rotor* reference axes are also connected rigidly to the stator and that the rotor harmonic refer-




(*a*) Rigidly connected to stator (*b*) Rigidly connected to rotor

FIG. 11.9. Non-holonomic reference frames.

ence axes also rotate with $2nv$ velocity with respect to the stator. This assumption simply groups the rotor meshes as shown in Fig. 11.9*a*.

2. However, it may also be assumed that the fundamental *stator* reference frame is rigidly connected to the rotor and that the harmonic stator frames rotate with $2nv$ with respect to the rotor. The new grouping is shown in Fig. 11.9*b*.

This last type of reference frame, in connection with the synchronous machine, was originally introduced by Blondel (the "two-reaction" theory). In dynamics both types are called "non-holonomic" reference frames.

## FREELY ROTATING REFERENCE FRAMES

Accepting the above two non-holonomic points of view, it is possible to assume that the fundamental reference axes are not connected rigidly to the stator, nor to the rotor, but that all rotate together freely at some arbitrary speed $v_f$, say at the speed of the impressed fluxes. By assuming that this transformation to the new speed was accomplished by means of a phase shifter with angle $\theta_f$, having been pushed through the whole network, *all absolute frequencies in the f meshes are increased by* $v_f$ *and in the* **b** *meshes by* $-v_f$ (see Fig. 9.5).

Of course, this change in the value of the speed terms is accomplished at the expense of the base frequency $f$, and so the sum of $f$ and all $v$ (namely, the absolute frequency) remains unchanged in each mesh.

Hence *the primitive equivalent circuit of Fig. 11.7 is valid for all four basic types of reference frames of Table 4.1.* The type of frame assumed determines only the groupings of the meshes (Figs. 11.8 and 11.9) and the groupings of the frequencies. *The equivalent circuit remains invariant under the four types of physical transformation of the reference frames.*

## THE "RULE OF SPEED" IN CROSSING THE AIRGAPS

In constructing time-harmonic equivalent circuits for single machines or for several interconnected machines, it is worthwhile to keep in mind the following rule, which is a special case of that given in Chapter 10 (Fig. 11.10):

In crossing the airgap from stator to rotor and back again, the absolute frequency increases (or decreases) by the rotor speed $v$ for each crossing. The zero speed reference occurs on the stator field side of the airgap, because of the excitation.

As a result, all stator harmonics are even products of $v$ and all rotor harmonics are odd products of $v$.

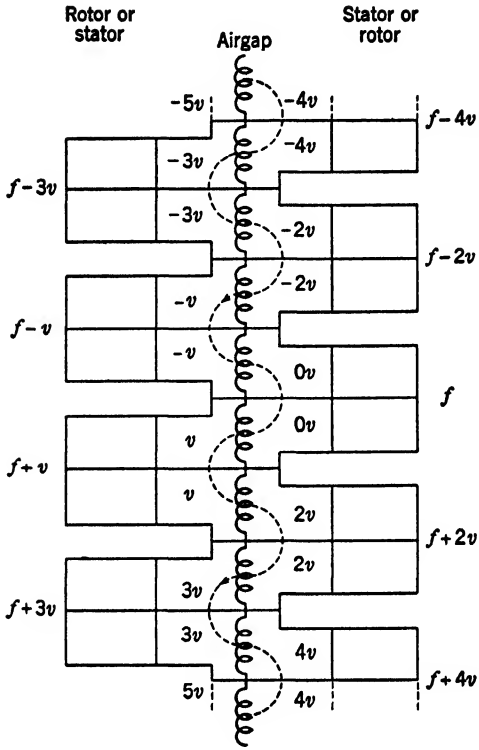



FIG. 11.10. The rule of $v$.

## UNBALANCED STATIONARY NETWORKS

Two-phase unbalanced resistors and inductors connected to the stator (or rotor), as in Fig. 5.8, follow the same representation as the stator (or rotor) resistances and leakage reactances. Hence, in Fig. 5.8, $f$ is replaced by the absolute frequency of the respective harmonic. The capacitors are divided always by the square of the absolute frequencies with which the resistances are divided.

## DOUBLY FED INDUCTION MOTOR WITH UNBALANCED STATOR AND ROTOR

Let an unbalanced load be added to the slip rings of an induction motor with asymmetrical stator. The stator voltages may be unbalanced. The equivalent circuit is shown in Fig. 11.11. When the rotor is also excited by an $f - v$ and an $f + v$ frequency forward voltages, the same network is used for both stator and rotor excitations, as shown. The stator and rotor impressed voltages match the frequencies of one network.

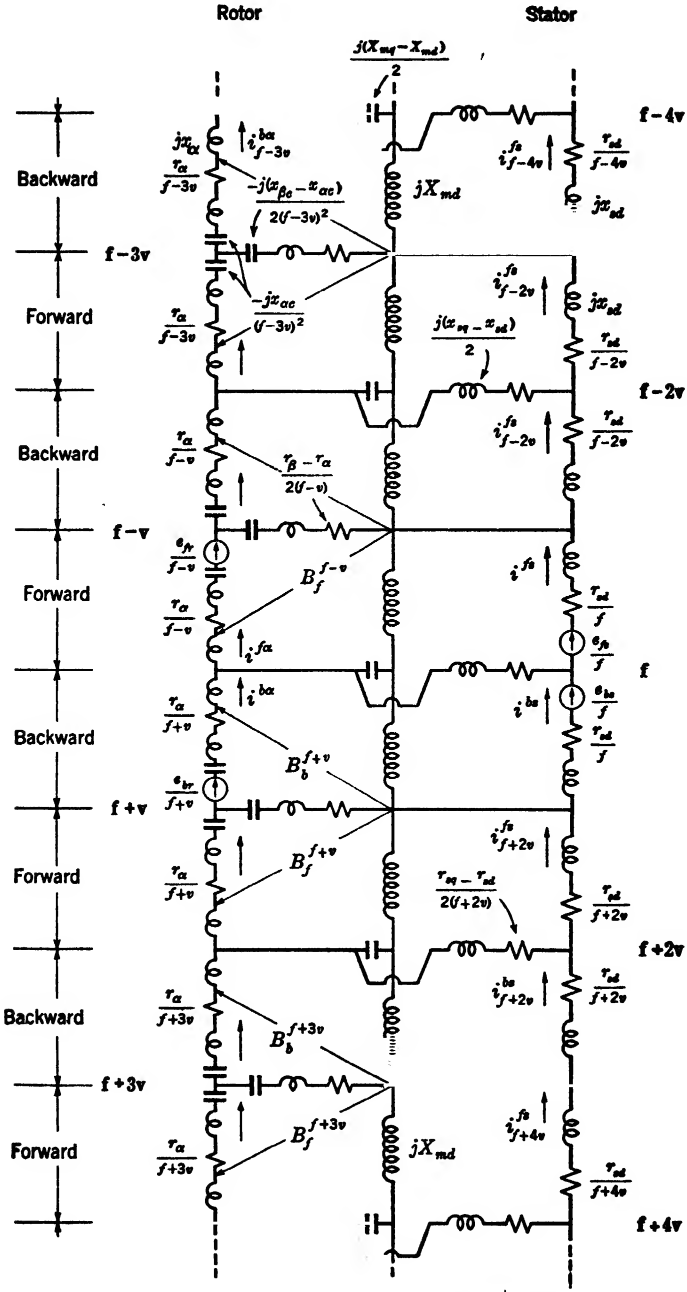



FIG. 11.11. Doubly fed induction motor with unbalanced stator and rotor loads.

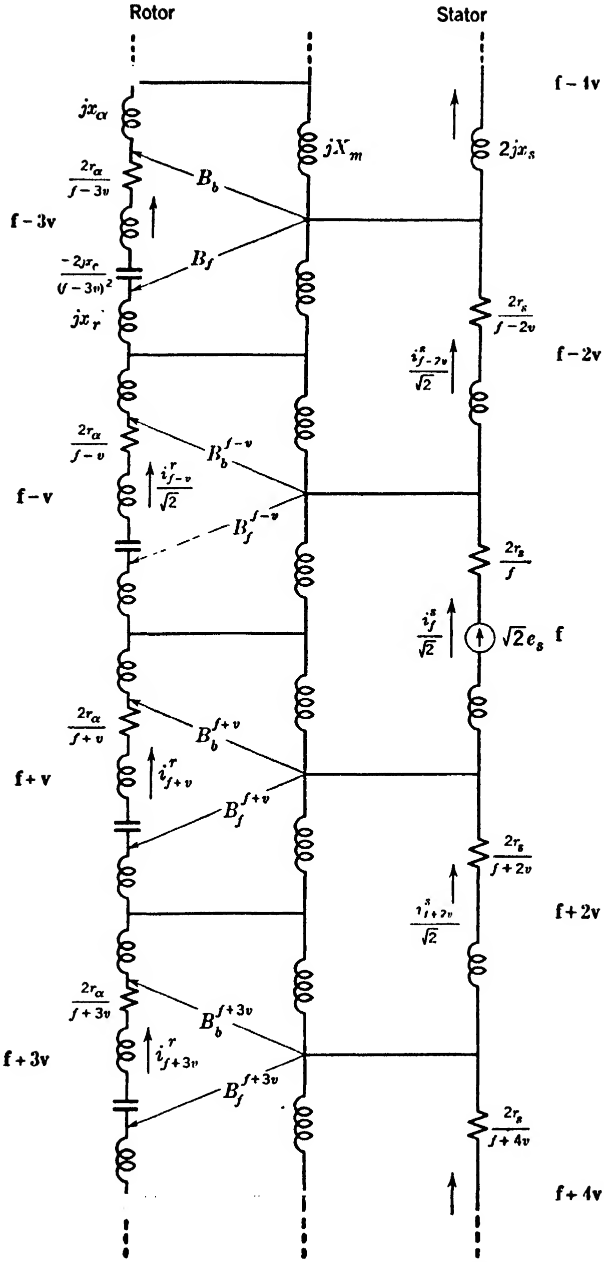



FIG. 11.12. Single-phase induction motor with single-phase rotor.

## SINGLE-PHASE INDUCTION MOTOR WITH SINGLE-PHASE ROTOR

In starting or reversing polyphase slip-ring induction motors, it occasionally happens that both stator and rotor are operated single-phase. Assuming the stator **q** axis and the rotor **β** axis windings open, *the common horizontal branches of Fig. 11.11 on the stator and rotor become open-circuited* (Fig. 11.12), since $ji^{qs}$ and $ji^{\beta r}$, which are now zero, flow

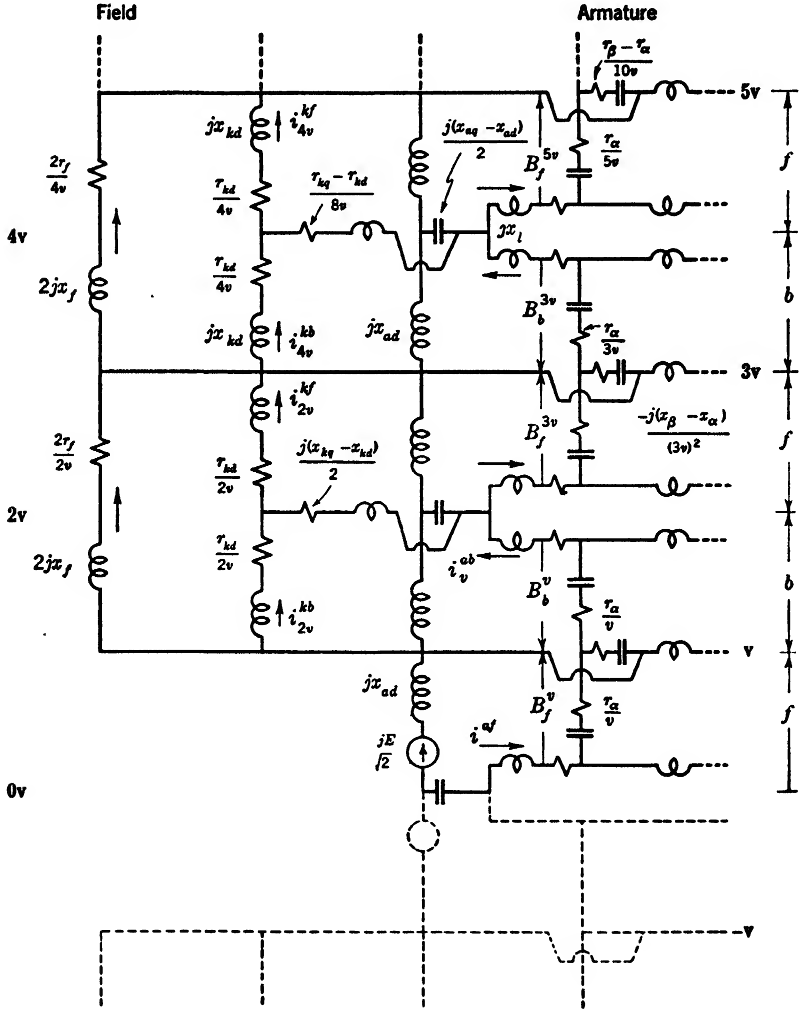


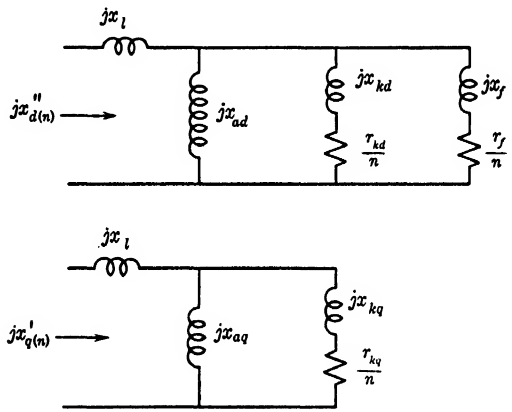

FIG. 11.13. Synchronous machine with unbalanced loads (excitation on the field only).

in the common branches. The stator and rotor harmonics are coupled through the airgap reactance $jX_m$.

For the calculation of torques the harmonic stator and rotor flux densities $B_{ds}$ and $B_{\alpha r}$ are found as the sum of the forward and backward flux densities $B_f$ and $B_b$ in each mesh. The open-circuit flux linkages $jB_{qs}$ and $jB_{\beta r}$, and hence the *open-circuit voltages,* are found by means of the difference between the $B_f$ and $B_b$ terms in each mesh.

## SYNCHRONOUS MACHINE WITH UNBALANCED LOADS

When the armature of a synchronous machine is connected to an unbalanced load, the network is given in Fig. 11.13. When the sole impressed voltage is d-c and is along the physical **d** axis, the currents




$n =$ ......$4v$, $2v$, 0, $-2v$, $-4v$..............if field is excited

$n =$ ......$f+3v$, $f+v$, $f-v$, $f-3v$......if armature is excited

FIG. 11.14. Operational impedances of field in the presence of time harmonics.

in the lower half of the network are conjugates of those in the upper half. Hence *the lower half of the network may be ignored,* as in all other synchronous machine networks having field excitation only (Fig. 6.12*c*).

In synchronous machine networks all the amortisseur and field meshes will be replaced by their operational impedances, as shown in Fig. 11.14. For each frequency a different $x''_d$ and $x'_q$ must be determined.

Indicating only short-circuit impedances, Fig. 11.15*a* shows the network when only the field is excited, whereas Fig. 11.15*b* shows the network when only the armature is excited.

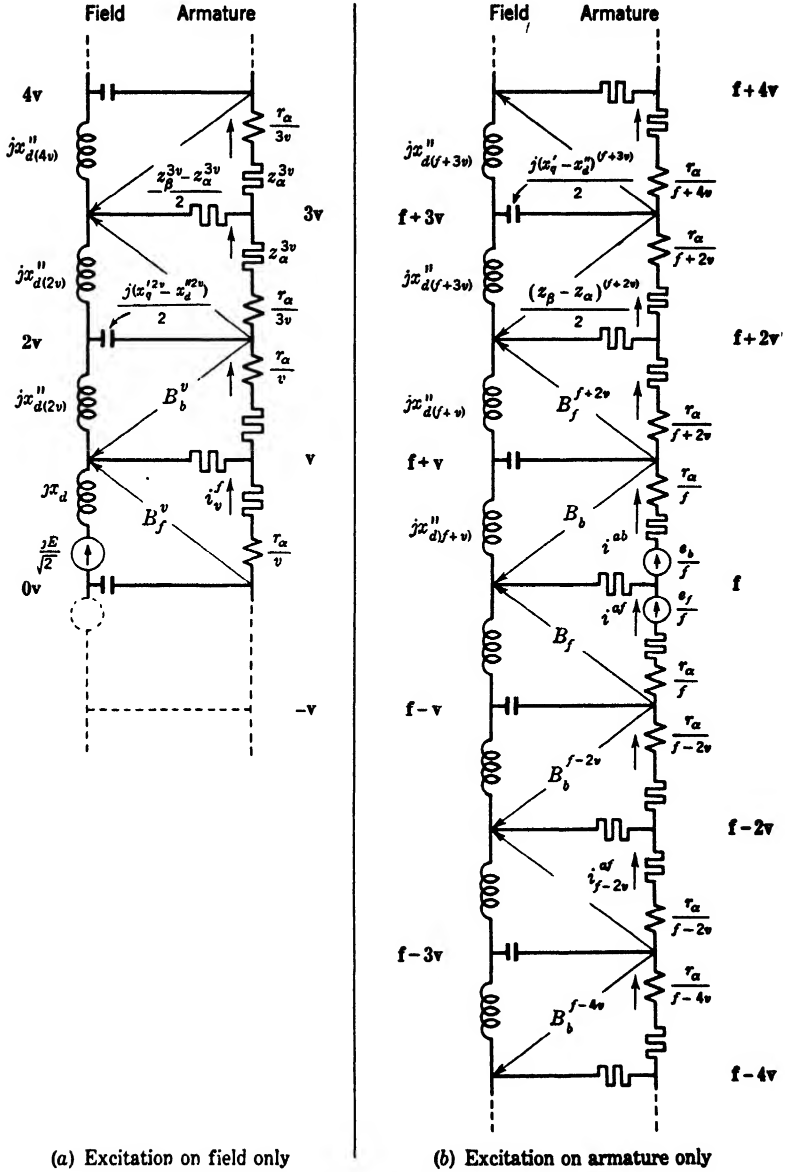



Fig. 11.15. Synchronous machine with unbalanced loads.

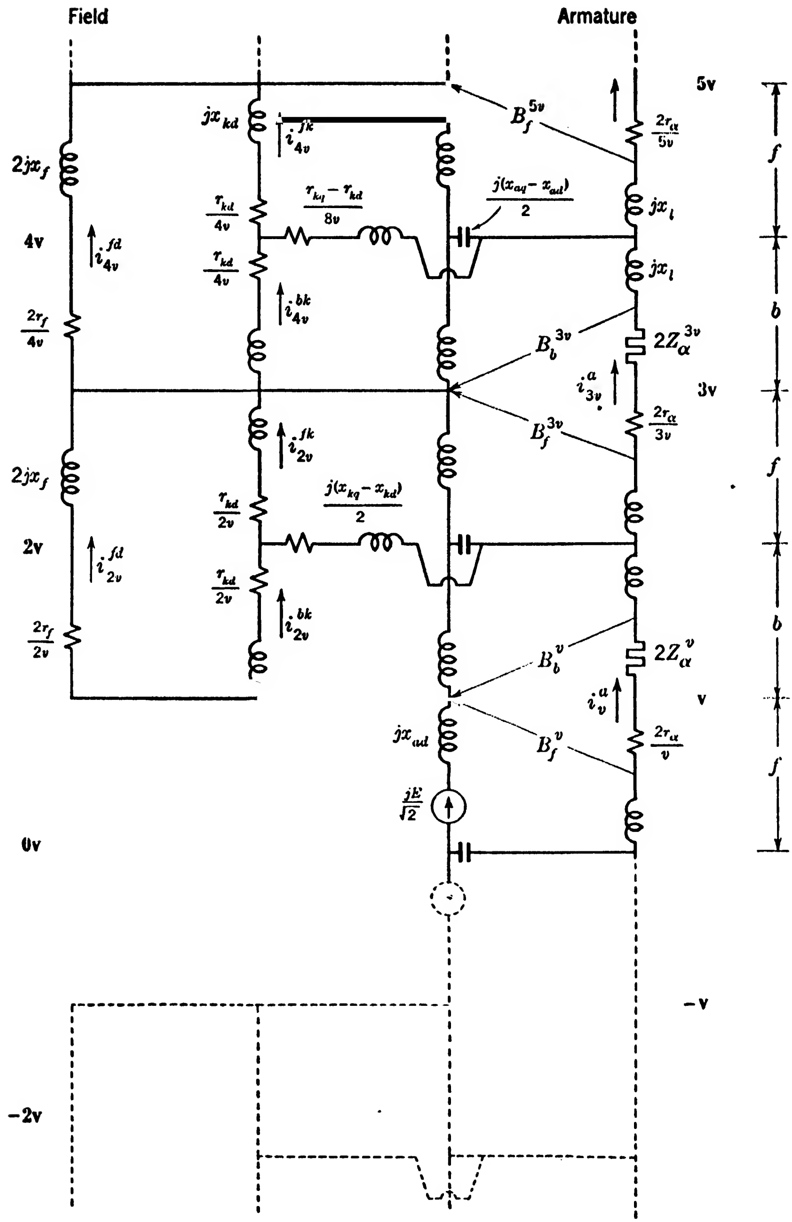



FIG. 11.16. Single-phase alternator.

## SINGLE-PHASE ALTERNATOR

A special case of Fig. 11.13 is the single-phase alternator (Fig. 11.16) supplying a single-phase load $Z_\alpha$. It is analogous to the single-phase induction motor with single-phase rotor (Fig. 11.12), except for a change in the absolute frequencies, since the base frequency $f$ of the voltage impressed on the field is zero. When the load $Z_\alpha$ is zero, the network corresponds to a sustained single-phase short circuit. When all field meshes are eliminated, the network is shown in Fig. 11.18*a*.

The open-circuit armature voltages across phase $\beta$ are found in terms of harmonics by means of the *differences* between the $B_f$ and $B_b$ terms in each mesh, which give the open-circuit flux linkages $jB_{\beta r}$ of phase $\beta$.

## SINGLE-PHASE SYNCHRONOUS MOTOR

A special case of Fig. 11.15*b* is the single-phase synchronous motor during starting, with only the armature excited (Figs. 11.17 and 11.18*b*). A numerical example is worked out at the end of this chapter.

## TORQUE CALCULATION OF SINGLE-PHASE MACHINES

In single-phase rotors and armatures the two flux densities $B_f$ and $B_b$, existing in a single mesh, may be combined into one flux across the center of the mesh, as shown in Fig. 11.18, representing only one half of $\sqrt{2}B_d = B_f + B_b$, namely $B_d/\sqrt{2}$. The substitution cannot be made in the mesh with the impressed voltage, unless the voltage is split into two. Hence *for the torque calculation twice the measured flux value has to be taken.*

By Eqs. 4.4 and 4.5 the torques are

$$T_+ = i^d B_d \qquad \text{and} \qquad T_- = i^{d*} B_d$$

Tables 11.1 and 11.2 are used for the single-phase alternator (Fig. 11.18*a*) (*with* $f = 0$) or for the single-phase induction motor (Fig. 11.12). (In the latter the diametral fluxes may be measured, if half the phase loads are taken twice for each mesh.) For a single-phase synchronous motor (Fig. 11.18*b*) Tables 11.3 and 11.4 are used.

## THE INTERCONNECTION OF TWO MACHINES WITH UNBALANCED STATOR AND ROTORS

When two machines, each with an unbalanced stator and rotor, are interconnected through their rotors (or armatures) and *both run at the same speed,* then their equivalent circuits are interconnected in the same manner as their prototype in Chapter 9, whose rotors (or armatures) were balanced. Now *variable phase shifters exist in the harmonic*

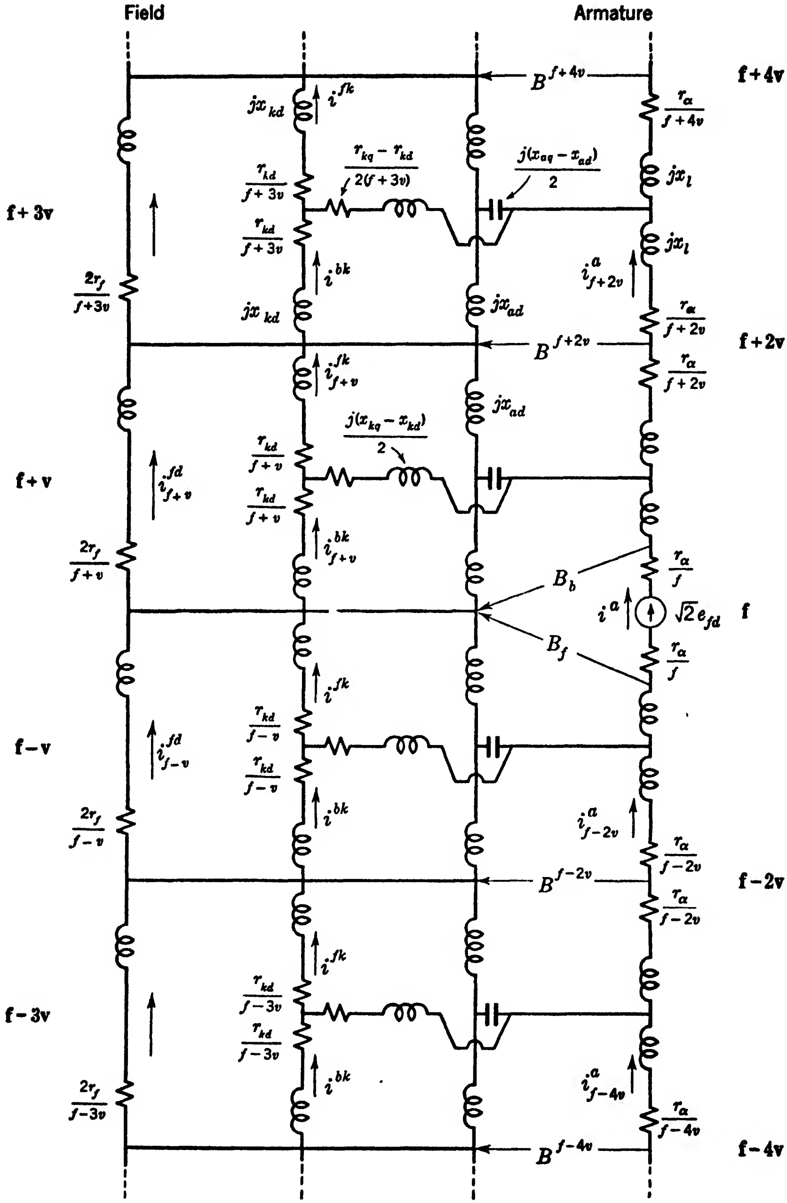
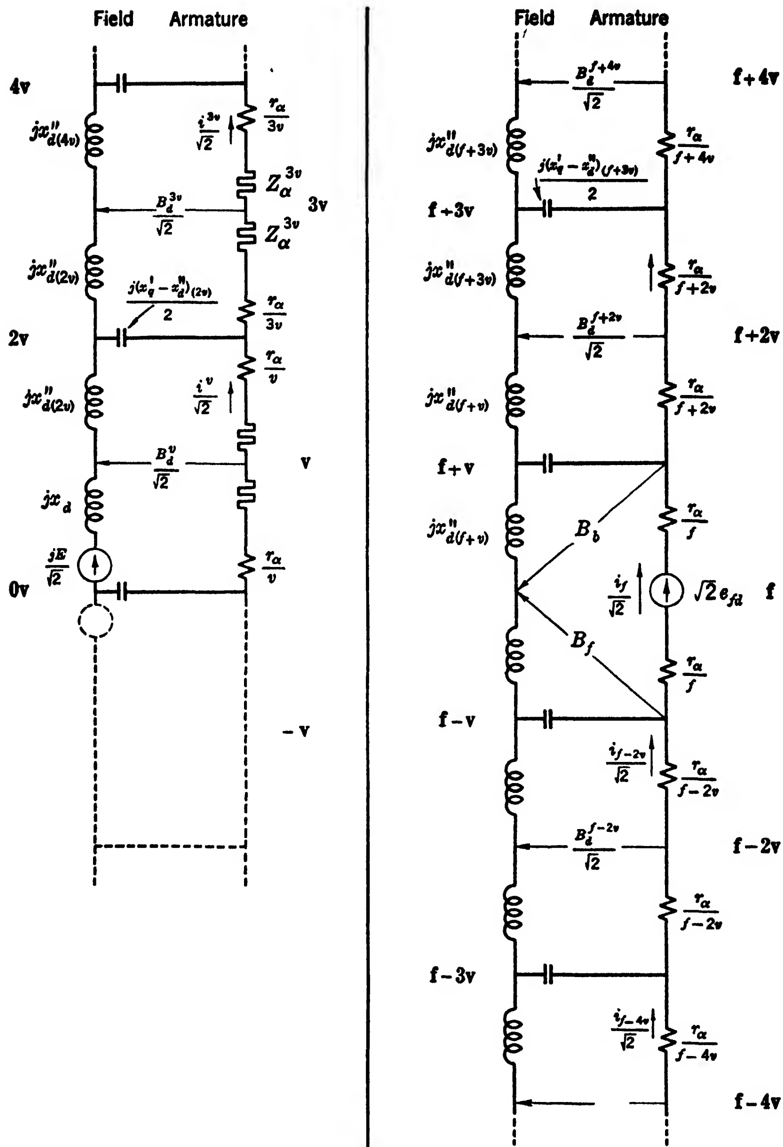



FIG. 11.17. Single-phase synchronous motor (starting conditions).




(*a*) Single-phase alternator (*b*) Single-phase synchronous motor

FIG. 11.18. Single-phase synchronous machines (field meshes eliminated).

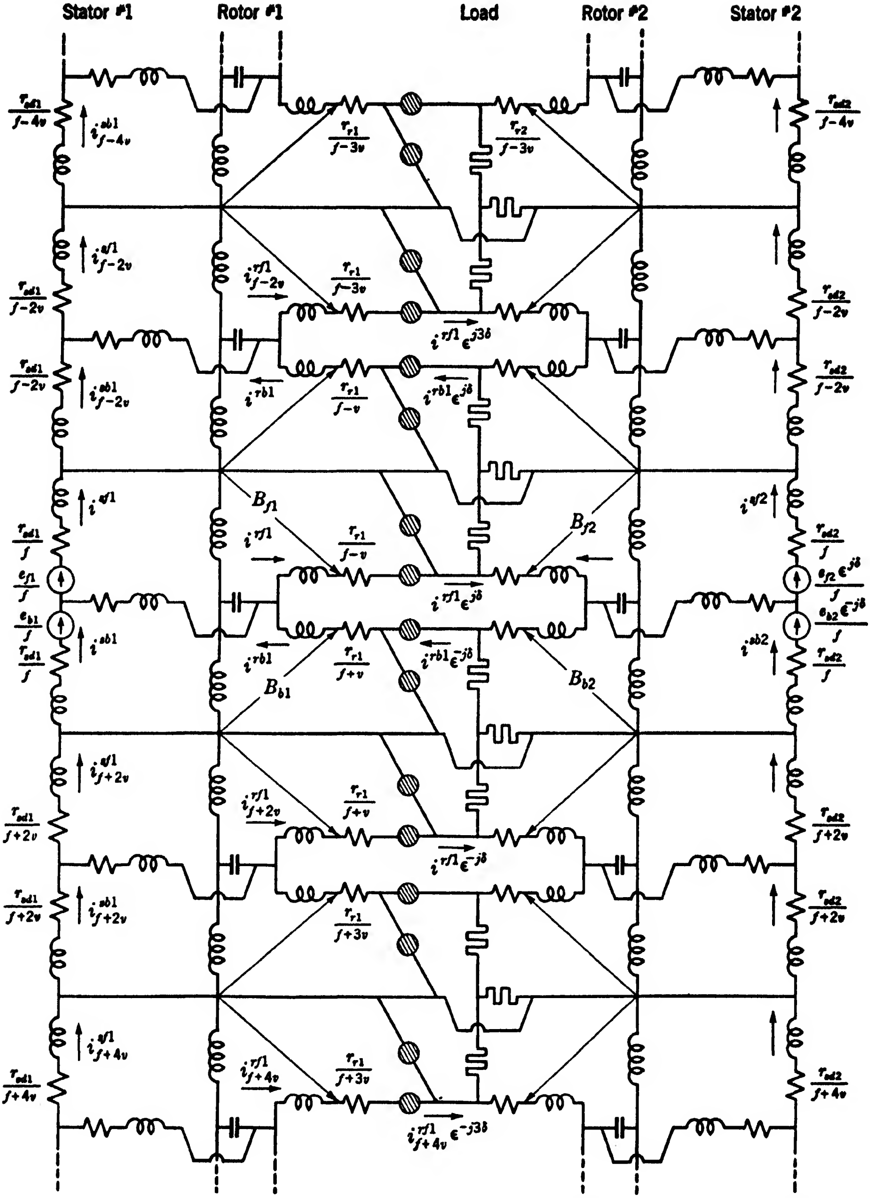



FIG. 11.19. Two unbalanced induction motors with unbalanced loads (running at the same speed).

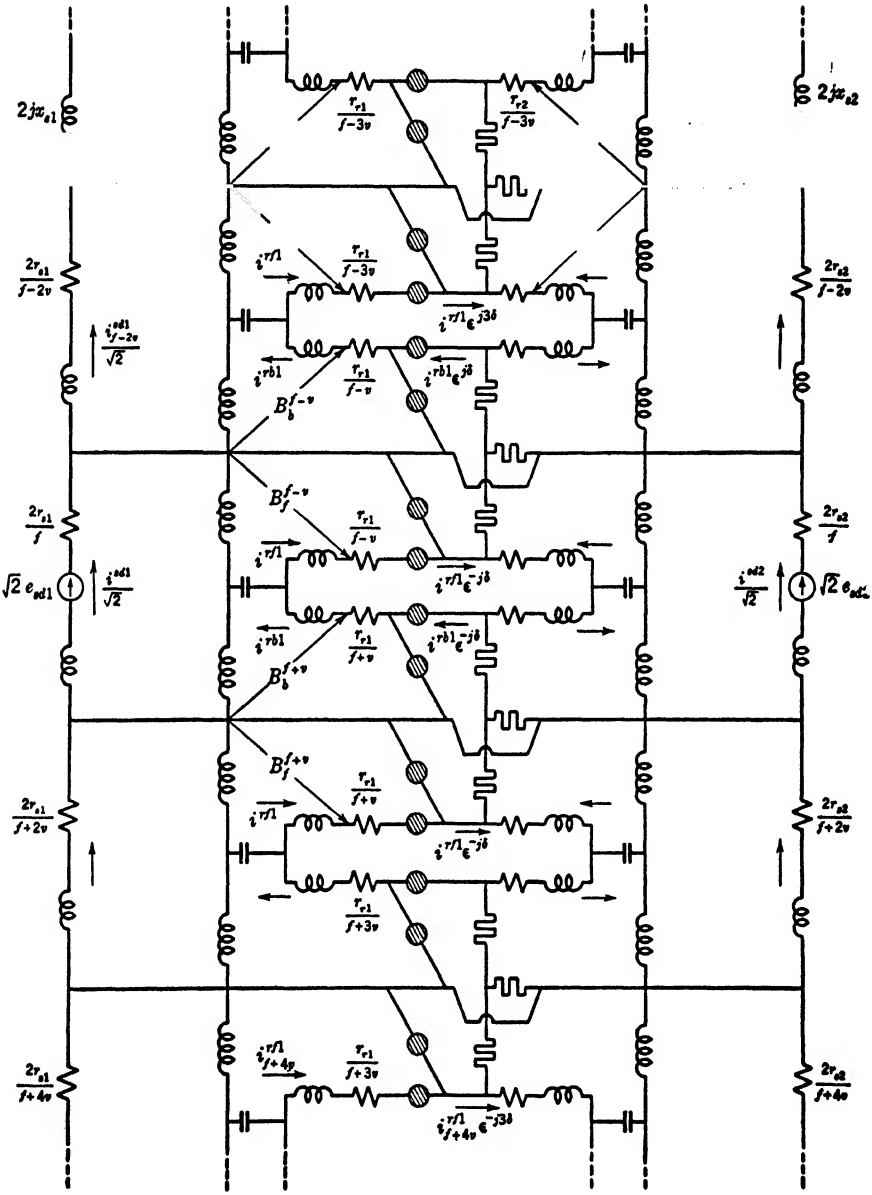



FIG. 11.20. Two single-phase selsyns with unbalanced loads (running at the same speed).

*meshes also*, containing $\theta$, $3\theta$, $5\theta$, in general $(1 + 2n)\theta$. They are combined into constant-angle phase shifters with $\delta = \theta_1 - \theta_2$, $3\delta = 3(\theta_1 - \theta_2)$, $5\delta$, etc., angles. The constant-angle phase shifters cannot be eliminated, however.

### TWO UNBALANCED INDUCTION MOTORS WITH UNBALANCED MOTOR LOADS

The equivalent circuit is given in Fig. 11.19, and its prototype with a balanced rotor load was given in Fig. 9.1*a*.

### TWO SINGLE-PHASE SELSYNS WITH UNBALANCED MOTOR LOADS

If the two set of unbalanced stator-impedance branches in Fig. 11.19 are open-circuited, the stators become single-phase (Fig. 11.20). The prototype with balanced rotor load was given in Fig. 9.11.

### TWO SYNCHRONOUS MACHINES WITH UNBALANCED LOADS

When the excitation in Fig. 11.20 becomes d-c, the circuit of Fig. 11.21 is the result, assuming the usual changes. The fields have been eliminated by the use of the short-circuit impedances. Again half the circuit is left out, as it contains only conjugate currents. The prototype with balanced load was given in Fig. 9.12.

### THE INTERCONNECTION OF TWO UNBALANCED MACHINES RUNNING AT DIFFERENT SPEEDS

It was found in Fig. 9.16 that, when two machines run at different speeds, only one of the stators may be unbalanced to avoid time harmonics. *When both stators are unbalanced, time harmonics appear even when the rotors are balanced.* In fact, in such a case any unbalance in the rotors starts an additional series of time harmonics. To avoid the extra complications, *the rotor loads will be assumed to be balanced.* Again, for each machine excitation, a separate network will have to be established.

Assuming only the first machine excited, *all its time harmonics originate in the second machine*, since its rotor is balanced. Hence the *various sets of time harmonic networks of the first machine are independent of each other* and are coupled together only through the second machine meshes. The second machine network is the standard time-harmonic network of Fig. 11.7, whereas the network of the first machine consists of several independent networks of Fig. 3.1, in each of which the absolute frequency assumes various harmonic values, as dictated by the corresponding coupled mesh of the second machine.

Although the various time harmonics of the first machine are isolated, nevertheless the "rule of speed" in crossing its airgap is still valid and its frequencies vary by $v_1$ for each crossing.

When the second machine only is excited, the same remarks apply, with the roles of the first and second machines interchanged.

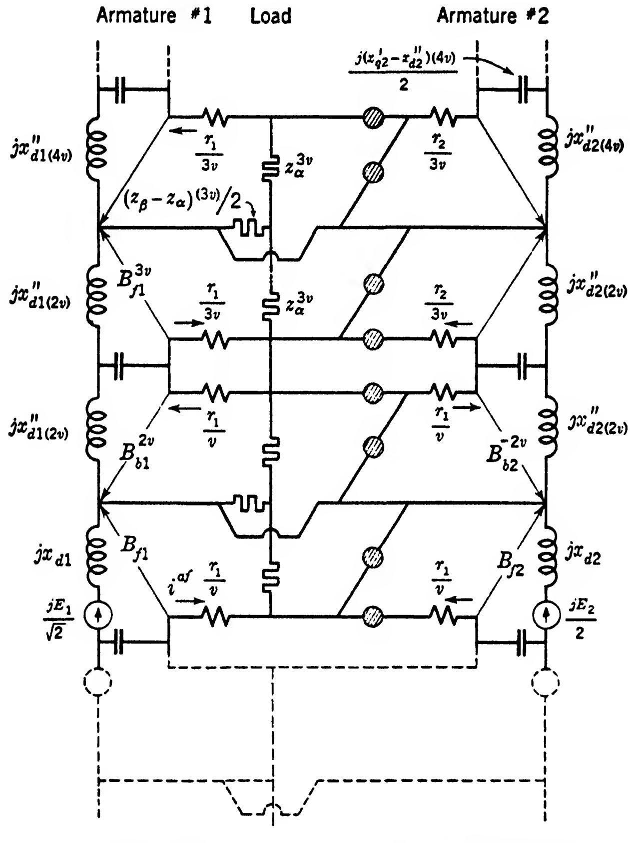



FIG. 11.21. Two synchronous machines with unbalanced loads (running at the same speed).

## TWO UNBALANCED INDUCTION MOTORS RUNNING AT DIFFERENT SPEEDS

Assuming only the first machine stator excited, the equivalent circuit is given in Fig. 11.22; the approximate circuit was shown in Fig. 9.16, and its prototype in Fig. 9.15. The harmonic torques of each machine are found by Eqs. 11.2 and 11.3.

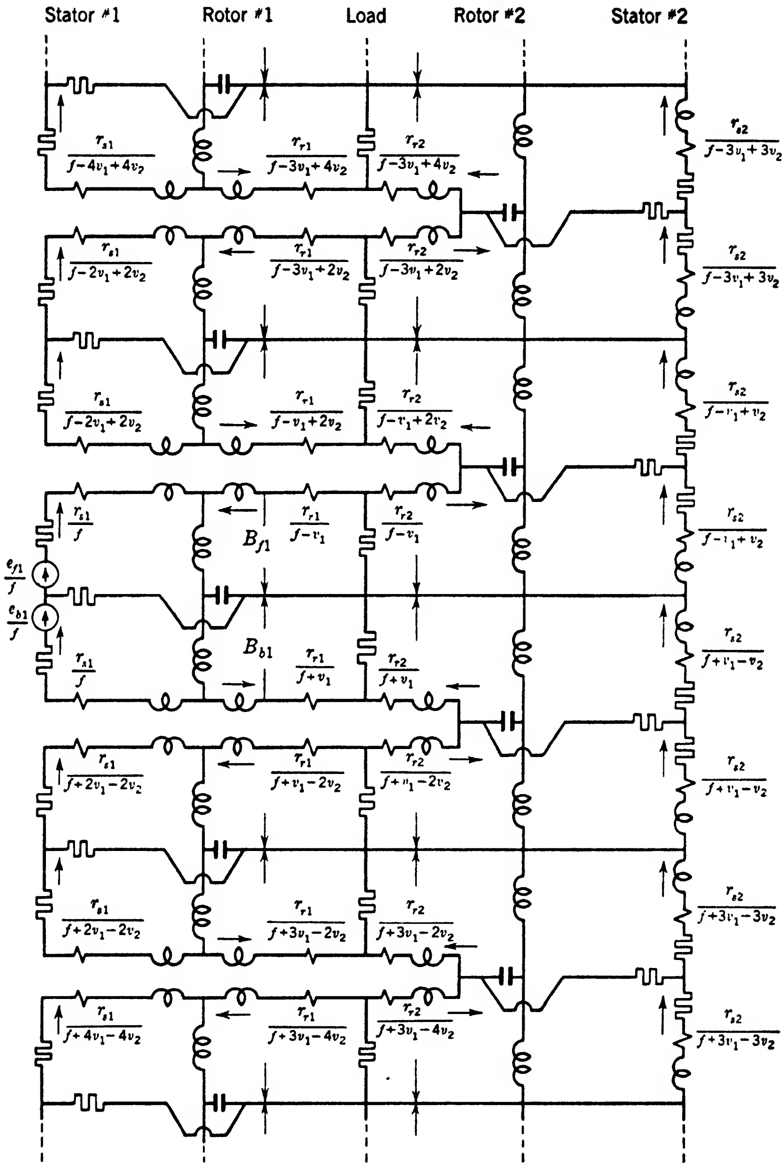



FIG. 11.22. Two unbalanced induction motors running at different speeds (excitation on first machine only).

## TWO SINGLE-PHASE SELSYNS RUNNING AT DIFFERENT SPEEDS

If the unbalanced stator impedances of Fig. 11.22 are open circuited, the stators become single-phase. The approximate circuit is given in Fig. 9.17. It should be remembered that *the roles of stator and rotor may be interchanged* and the single-phase salient structure may be the rotating member.

## TWO SYNCHRONOUS MACHINES RUNNING AT DIFFERENT SPEEDS

During starting, running, and pulling into step, two salient-pole synchronous machines often may run at different speeds, while the field of each is excited by a d-c source. The network is shown in Figs. 11.23

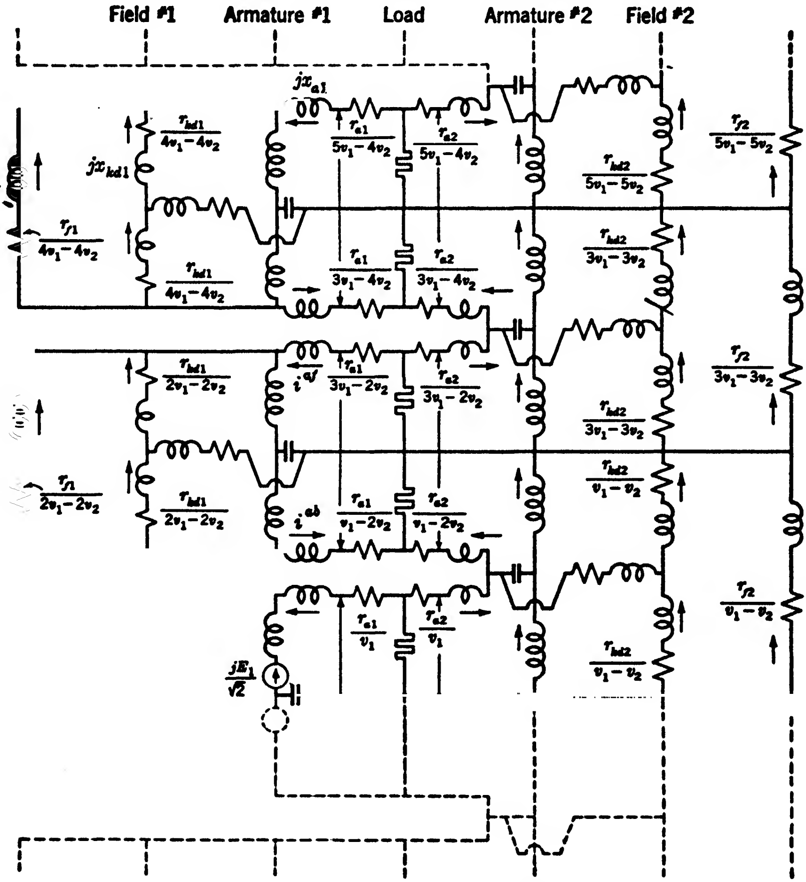



FIG. 11.23. Two synchronous machines running at different speeds (field excitation on first machine only).

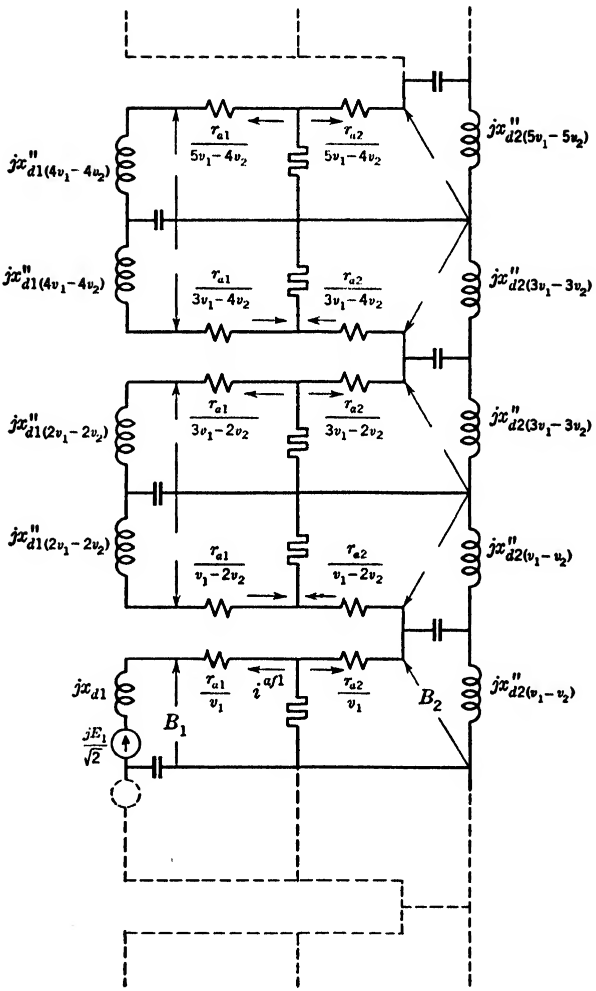



FIG. 11.24. Two synchronous machines running at different speeds (field meshes eliminated from Fig. 11.23).

and 11.24, and its approximate circuit is given in Fig. 9.18. For each harmonic a different $x''_{dn}$ and $x'_{q(n)}$ have to be calculated, as shown in Fig. 11.14.

## NUMERICAL EXAMPLE OF A SINGLE-PHASE SYNCHRONOUS MOTOR

The three-phase constants of a 5000-horsepower, 25-cycle, single-phase synchronous motor, corresponding to Fig. 6.26 (or to Fig. 6.7 at standstill), are given in Fig. 11.25, on a 5000-kva, 1550-volt base.

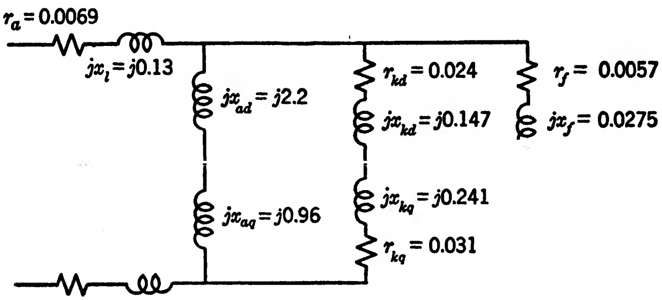



FIG. 11.25. Design constants of a single-phase synchronous motor.

The equivalent circuit of the motor under starting conditions is given in Fig. 11.17, or, if the field meshes are eliminated, in Fig. 11.18*b*. Assuming only *three* meshes in the last-mentioned circuit, near synchronous speed at $v = 0.9$ (or $s = 0.1$) the operational impedances $x''_d$ and $x'_q$ for the various harmonics are given in Table 11.5.

TABLE 11.5 HARMONIC OPERATIONAL IMPEDANCES AT $S = 0.1$

| | |
|---|---|
| $x''_{d(1-3v)} = -0.0058725 + j0.22196$ | $x'_{q(1-3v)} = -0.011648 + j0.32282$ |
| $x''_{d(1-v)} = -0.078693 + j0.25158$ | $x'_{q(1-v)} = -0.18569 + j0.37057$ |
| $x''_{d(1+v)} = 0.0052555 + j0.22192$ | $x'_{q(1+v)} = 0.010423 + j0.32278$ |
| $x''_{d(1+3v)} = 0.0027006 + j0.22183$ | $x'_{q(1+3v)} = 0.005353 + j0.32268$ |

The resulting three-mesh network is given in Fig. 11.26*a*, where all impedances in a branch have been combined. Elimination of the upper and lower meshes give Fig. 11.26*b*, which is reduced to one impedance in Fig. 11.26*c*. Finding the fundamental-frequency ($f$) current $i^f$ in the last figure, the $f + 2v$ frequency currents can be calculated from Fig. 11.26*a*. They are shown in Fig. 11.26*d*.

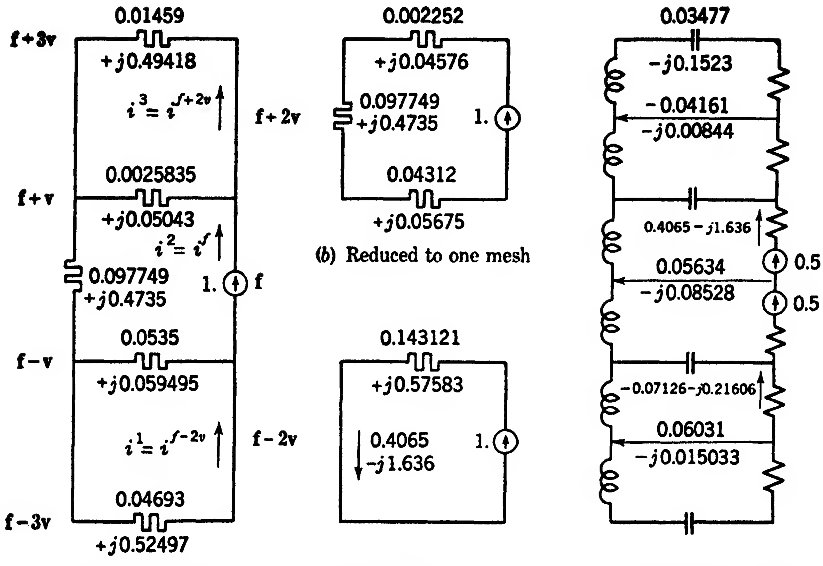



FIG. 11.26. Solution of network at $s = 0.1$.

Knowing the impedances and currents in Fig. 11.26*a*, the differences of potential give the three diametral fluxes, as shown in Fig. 11.26*d*.

The formulas and calculated values for the various harmonic torques are shown in Table 11.6.

TABLE 11.6 FUNDAMENTAL AND HARMONIC TORQUES AT $S = 0.1$

| | | |
|---|---|---|
| $T_0 \quad = \text{Real of } 2(i^*{}_1B_1 + i^*{}_2B_2 + i^*{}_3B_3)$ | = | 0.3224 |
| $T_{2v} \quad = 2(i^*{}_1B_2 + i^*{}_2B_3 + i_2B^*{}_2 + i_3B^*{}_2)$ | = | $0.106436 \quad - j0.302822$ |
| $T_{4v} \quad = 2(i^*{}_1B_3 + i_3B^*{}_1)$ | = | $0.018346 \quad - j0.03409$ |
| $T_2 \quad = 2(i_1B_3 + i_3B_1 + i_2B_2)$ | = | $-0.23133 \quad - j0.25388$ |
| $T_{2+2v} = 2(i_2B_3 + i_3B_2)$ | = | $-0.08348 \quad + j0.133158$ |
| $T_{2-2v} = 2(i_1B_2 + i_2B_1)$ | = | $-0.04502 \quad - j0.221752$ |
| $T_{2+4v} = 2i_3B_3$ | = | $0.0054628 + j0.01208$ |
| $T_{2-4v} = 2i_1B_1$ | = | $0.0150914 - j0.2392$ |

The fundamental and some of the harmonic torques (peak values) are plotted on Fig. 11.27 from $v = 0.8$ to $v = 1$.

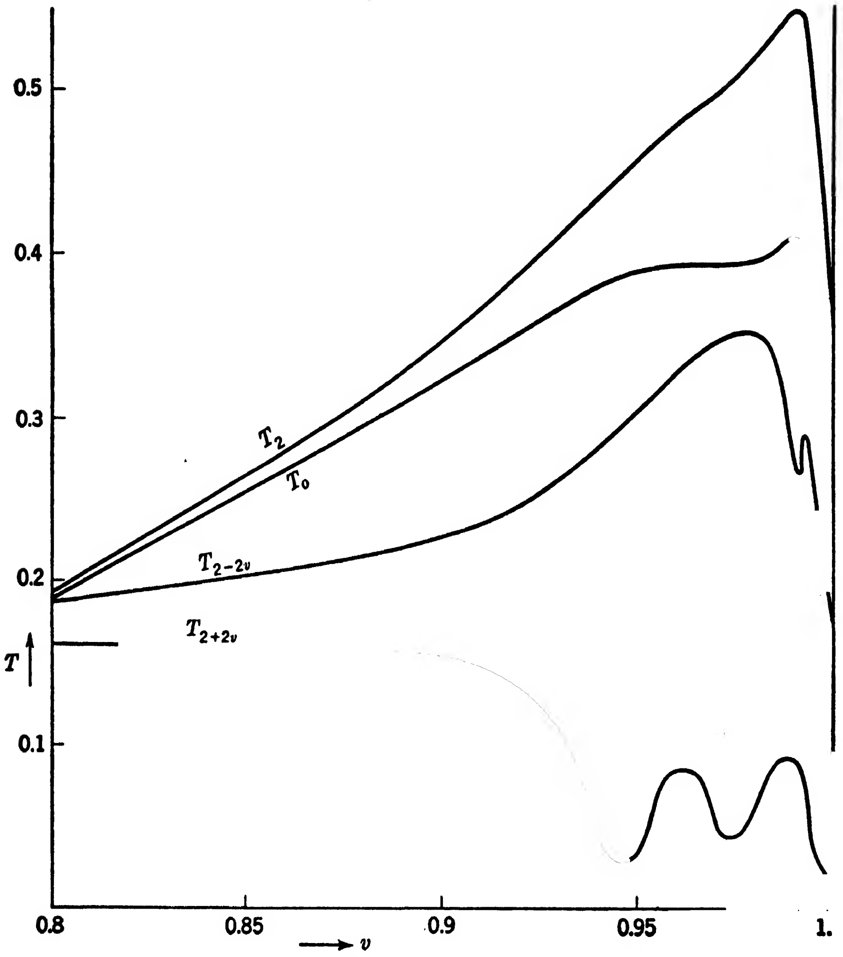



FIG. 11.27. Fundamental and some of the harmonic torques during starting of a single-phase synchronous motor.

# 12 SUDDEN SHORT CIRCUITS AND LOAD VARIATIONS

## THE "CONSTANT FLUX-LINKAGE" THEOREM

The calculation of sudden short circuits and sudden load variations of rotating machines often involves the solution of linear differential equations with variable coefficients. To avoid the mathematical difficulties inherent in such problems, engineers have introduced several approximate methods of solution which nevertheless lead to satisfactory answers. One of the most valuable assumptions is the "constant flux-linkage" theorem. For the present purpose it may be stated as follows:

*At the instant of short circuit (or load variation) all fluxes linking each winding become frozen in magnitude.* As the rotor keeps on rotating, the currents in every winding adjust themselves in such a manner as to maintain temporarily the flux linkages in each winding constant. For purposes of analysis the flux linkages are maintained constant during the first few revolutions.

During the actual short circuit (or load variation) the fluxes appearing during the first cycle do not remain constant but decrease in value and eventually settle down to their short circuit (or final load) values.

## THE DURATION OF SUDDEN LOAD VARIATIONS

When the sustained short-circuit fluxes are not d-c, the above "constant flux-linkage" theorem refers only to a portion of the fluxes. To describe the history of the fluxes during the short circuit, the duration of the phenomenon can be divided into *four* time periods:

1. The instant of short circuit.
2. The first few cycles (revolution) following it.

3. One or more particular cycles (revolutions) picked out sometime later.

4. The sustained short circuit (or new load condition), after all transients have died out.

In the following *a reference to the phenomenon of sudden load variation also will be understood, when only a sudden short circuit is referred to.*

## THE MECHANISM OF SUDDEN SHORT CIRCUITS

At the instant of short circuit a definite amount of flux links each winding. Each of the fluxes is subdivided at the instant of short circuit into two parts:

1. The final sustained short-circuit flux that would exist at that instant (under the same condition of operation), if the transients had already died out.

2. The difference between the actually existing total flux and the final flux.

*These two fluxes behave differently during the first few instants following the short circuits.* In particular:

1. The final, sustained flux varies in exactly the same manner as if no difference flux had existed. Specifically, it may keep on rotating (for instance, in the stator of an induction motor), or it may remain constant and stationary with respect to the winding (for instance, in the field of a synchronous machine), or it may vary in magnitude and space position.

2. The difference between the initial and final flux, existing in each mesh of the machine the instant of short circuit, becomes a "trapped" flux regardless of whether the mesh is shorted or left unchanged. *It becomes a d-c flux with respect to the iron structure.* (In a commutator machine it becomes a d-c flux with respect to the *brushes.*) Extra currents must appear in each winding in order to maintain these d-c fluxes.

The assumption that the trapped flux stops its rotation (if any) and becomes a d-c flux for the next few instants is an approximation based on the "constant flux-linkage" theorem.

During the transient period the final sustained flux maintains its steady-state rate, just as in the first few instants after the short circuit. However, *the trapped d-c fluxes decrease exponentially in magnitude* (still remaining stationary with respect to the iron structure) and eventually disappear. This decrease is given by the "decrement factor," which is different for each trapped flux.

The calculation of the decrement factor is given at the end of this chapter.

## THE TWO TYPES OF EQUIVALENT CIRCUITS

Two different equivalent circuits must be established to represent the two types of system conditions that play a part in the approximate calculations. In particular:

1. The performance of the machine before the short circuit is given by the so-called "pre-short-circuit" network. (Before the short circuit the machine terminals may be open-circuited.) Accordingly, the pre-short-circuit network may be a simple open-circuited network, or it may be the general unbalanced network with the time-harmonic meshes. All impressed voltages existing before the short circuit are known.

The magnitude of the flux densities $B$ of each winding, to be needed later on, is given by the differences of potentials $B$ measured across the inductors. The *absolute frequency* of the flux densities (namely, the frequency of fluxes with respect to the conductors) is given by the denominator of the respective winding resistance.

2. The effect of the short circuit or the load variation is to introduce a new equivalent circuit, the so-called "post-short-circuit" network, in which the loads or the open-circuit terminals are short-circuited or are replaced by another type of load. With an unbalanced load or with a single-phase short circuit, the new equivalent circuit is the general unbalanced network containing time harmonics.

## THE TWO TYPES OF "POST-SHORT-CIRCUIT" PERFORMANCE

On this post-short-circuit equivalent circuit two types of voltages are impressed:

1. The known impressed voltages existing on the machine *after* the short circuit or load variation. They give the needed *sustained short circuit* performance, and they also give those portions of the initial fluxes that are not part of the trapped d-c fluxes.

2. *The trapped flux densities $B$ measured on the pre-short network* (and the sustained short-circuit network) *are used as impressed voltages on the "post-short-circuit" network,* to find the transient currents that maintain the trapped fluxes constant.

Although both the pre-short-circuit and the sustained short-circuit performances under known impressed voltages have already been treated hitherto, *the performance under the impressed d-c fluxes that are trapped in each winding require further explanations. The rest of the chapter will deal only with the trapped fluxes,* with the transient currents necessary to maintain them, and with the transient torques due to their existence.

## THE SERIES OF STEADY-STATE PHENOMENA

When a machine is suddenly short-circuited, several revolutions of the rotor take place before the currents settle down to their final sustained short-circuit values. The equivalent circuit solution divides this transient phenomenon into a *series of steady-state phenomena*, each steady-state lasting at least through one cycle (revolution) during which the trapped flux remains constant. The values of the currents and torques during this one particular cycle are found by the equivalent circuit as the sums of time harmonics. The succession of steady-state phenomena to be calculated is the following:

1. The *initial* short-circuit currents and torques are calculated by using the actual flux densities $B$ appearing in the pre-short-circuit (and post-short-circuit) network as impressed voltages.
2. The *intermediate* currents and torques, at any particular time after the short circuit, are calculated by first finding the decreased values of the trapped fluxes at the particular time desired and then using them as impressed voltages.

*The sum of either of these currents and the sustained* or final short-circuit *currents* is the actual currents existing during the particular revolution under consideration. The torques are due to the actual currents and fluxes.

The actual values of the short-circuit currents and torques assumed in the transition period are pieced together from these calculated steady-state values.

## SUDDEN SHORT CIRCUIT OF SYNCHRONOUS MACHINES

When the armature of a synchronous machine is short-circuited (polyphase or single-phase short circuit), during the *sustained* short circuit the armature flux is zero, while the field flux is d-c and it rotates with the field structure.

Since the sustained field flux behaves also as a trapped flux, *there is no need to divide into two parts the field fluxes at the instant of short circuit.* All the field flux may be treated as a trapped d-c flux, which decreases, however, not to zero, but to a constant d-c value.

Hence for the sudden armature short-circuit calculation of a synchronous machine the original simple definition of the "constant flux-linkage" theorem suffices. However, when the armature load changes from one value to some other value, the sustained value of the armature flux is finite and sinusoidal with respect to the armature. Consequently, the armature flux must be divided into two parts.

## "TRAPPED" FLUX-LINKAGES AS IMPRESSED VOLTAGES

*Before* the short circuit, the frequency of the flux $B$ of each winding with respect to the conductors (the absolute frequency) is usually different for each winding. *After the short circuit the absolute frequency of each trapped $B$ becomes zero.*

If the pre-short-circuit network contains no time harmonics, then only two networks will have to be established to impress all the trapped fluxes; in particular, one for all the *field* (or stator) fluxes and one for all the *armature* (or rotor) fluxes. *The base frequency $f$ for each case becomes zero.*

## THE DISAPPEARANCE OF RESISTANCES IN D-C MESHES

Since the absolute frequency of the impressed voltage is zero (d-c), the resistance in series with the impressed voltage becomes infinite. The situation is similar to that existing in the field of a synchronous machine (Fig. 6.12), where also a d-c voltage is impressed. *The resistances in series with the d-c voltages disappear in both cases* (but not the resistances in the other meshes).

However, in the standard synchronous machine (Fig. 6.12), the value of the current $i^{fd}$ flowing is known, whereas in short-circuit problems the resultant flux $B$ linking the winding is known. Hence an impressed voltage $B$ will exist now in a d-c mesh, instead of an impressed current. A similar situation existed in Fig. 6.13, where the constant field flux $\sqrt{2}jB_{fq}$ is impressed on the field.

## TORQUE CALCULATIONS FOR SUDDEN SHORT CIRCUITS

The torque formulas given in Tables 11.1 to 11.4 assumed that either the stator *alone* or the rotor alone is excited.

In sudden short-circuit problems *both stator and rotor are excited simultaneously, each producing a set of time harmonics that differ in frequencies.* Hence for this torque calculation two sets of rotor $i$-s and $B$-s have to be considered, each produced by a different set of impressed voltages.

The rotor forward (or backward) currents are

$$i = \cdots i^{-4v} + i^{-2v} + i^0 + i^{2v} + i^{4v} + \cdots + \cdots i^{-3v} + i^{-v} + i^v + i^{3v} + \cdots \qquad 12.1$$

The rotor fluxes are of the same form. Hence the sum-frequency torques are, 4.7, by Eq.

$$T_+ = i^f B_b + i^b B_f$$

Leaving out the indices $f$ and $b$,

TABLE 12.1 SUM-FREQUENCY TORQUES

| $iB =$ | $B^{3v}$ | $B^{2v}$ | $B^{v}$ | $B^{0}$ | $B^{-v}$ | $B^{-2v}$ |
|---|---|---|---|---|---|---|
| $i^{3v}$ | $6v$ | $5v$ | $4v$ | $3v$ | $2v$ | $v$ |
| $i^{2v}$ | $5v$ | $4v$ | $3v$ | $2v$ | $v$ | 0 |
| $i^{v}$ | $4v$ | $3v$ | $2v$ | $v$ | 0 | $-v$ |
| $i^{0}$ | $3v$ | $2v$ | $v$ | 0 | $-v$ | $-2v$ |
| $i^{-v}$ | $2v$ | $v$ | 0 | $-v$ | $-2v$ | $-3v$ |
| $i^{-2v}$ | $v$ | 0 | $-v$ | $-2v$ | $-3v$ | $-4v$ |

Two such sets of torques have to be established.

The difference-frequency torques are, by Eq. 4.8,

$$T_- = i^{f*}B_f + i^{b*}B_b \qquad 12.2$$

Leaving out the indices $f$ and $b$,

TABLE 12.2 DIFFERENCE-FREQUENCY TORQUES

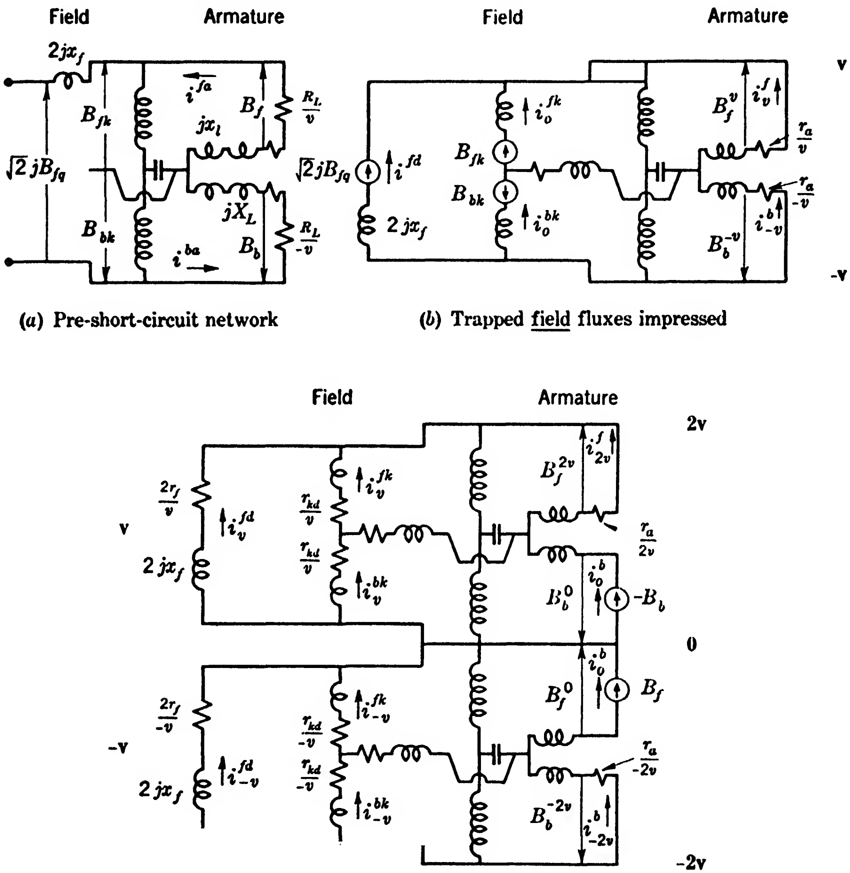
| $i^*B =$ | $B^{3v}$ | $B^{2v}$ | $B^{v}$ | $B^{0}$ | $B^{-v}$ | $B^{-2v}$ |
|---|---|---|---|---|---|---|
| $i^{*3v}$ | 0 | $-v$ | $-2v$ | $-3v$ | $-4v$ | $-5v$ |
| $i^{*2v}$ | $v$ | 0 | $-v$ | $-2v$ | $-3v$ | $-4v$ |
| $i^{*v}$ | $2v$ | $v$ | 0 | $-v$ | $-2v$ | $-3v$ |
| $i^{*0}$ | $3v$ | $2v$ | $v$ | 0 | $-v$ | $-2v$ |
| $i^{*-v}$ | $4v$ | $3v$ | $2v$ | $v$ | 0 | $-v$ |
| $i^{*-2v}$ | $5v$ | $4v$ | $3v$ | $2v$ | $v$ | 0 |

Again two such sets of torques have to be established.

## THREE-PHASE SHORT CIRCUIT OF A BALANCED ALTERNATOR

When a synchronous machine supplying a balanced load is subjected to a *three-phase* short-circuit, neither the pre-short-circuit nor the post-short-circuit network contains any higher time harmonics. The transient

phenomenon due to the trapped fluxes is equivalent to a synchronous machine with balanced armature, excited simultaneously from the field and from the armature by zero frequency voltages (Fig. 12.1). The




(*a*) Pre-short-circuit network

(*b*) Trapped field fluxes impressed

(*c*) Trapped armature fluxes impressed

FIG. 12.1. Three-phase short circuit of an alternator.

network for the armature fluxes is identical with Fig. 11.5, except for the frequency.

For the *first cycle* after the short circuit, the impressed flux densities $B$ are read off the pre-short-circuit network. (Since the sustained field flux is also d-c, it need not be divided into two components, as has been mentioned above.) For any *subsequent cycle* the impressed $B$ have to be calculated by the method to be shown later.

The sustained short-circuit currents are found from Fig. 6.5*a*.

## THREE-PHASE SHORT-CIRCUIT TORQUE CALCULATIONS

In finding the short-circuit torques, three forward and three backward armature currents and fluxes are available from Figs. 12.1*b* and *c*. In particular:

$$\text{Forward:}\quad i_0{}^f,\quad i_v{}^f,\quad i_{2v}{}^f,\quad B_f{}^0,\quad B_f{}^v,\quad B_f{}^{2v}$$

$$\text{Backward:}\quad i_0{}^b,\quad i_{-v}{}^b,\quad i_{-2v}{}^b,\quad B_b{}^0,\quad B_b{}^{-v},\quad B_b{}^{-2v}$$

The sum-frequency torques are, by Table. 12.2,

TABLE 12.3 SUM-FREQUENCY TORQUES

$T_+ =$

| | $B_b{}^0$ | $B_b{}^{-v}$ | $B_b{}^{-2v}$ |
|---|---|---|---|
| $i_{2v}{}^f$ | $2v$ | $v$ | $0$ |
| $i_v{}^f$ | $v$ | $0$ | $-v$ |
| $i_0{}^f$ | $0$ | $-v$ | $-2v$ |

$+$

| | $B_f{}^{2v}$ | $B_f{}^v$ | $B_f{}^0$ |
|---|---|---|---|
| $i_0{}^b$ | $2v$ | $v$ | $0$ |
| $i_{-v}{}^b$ | $v$ | $0$ | $-v$ |
| $i_{-2v}{}^b$ | $0$ | $-v$ | $-2v$ |

The difference-frequency torques are, by Table 12.2,

TABLE 12.4 DIFFERENCE-FREQUENCY TORQUES

$T_- =$

| | $B_f{}^{2v}$ | $B_f{}^v$ | $B_f{}^0$ |
|---|---|---|---|
| $i_{2v}{}^{f*}$ | $0$ | $v$ | $2v$ |
| $i_v{}^{f*}$ | $-v$ | $0$ | $v$ |
| $i_0{}^{f*}$ | $-2v$ | $-v$ | $0$ |

$+$

| | $B_b{}^0$ | $B_b{}^{-v}$ | $B_b{}^{-2v}$ |
|---|---|---|---|
| $i_0{}^{b*}$ | $0$ | $-v$ | $-2v$ |
| $i_{-v}{}^{b*}$ | $v$ | $0$ | $-v$ |
| $i_{-2v}{}^{b*}$ | $2v$ | $v$ | $0$ |

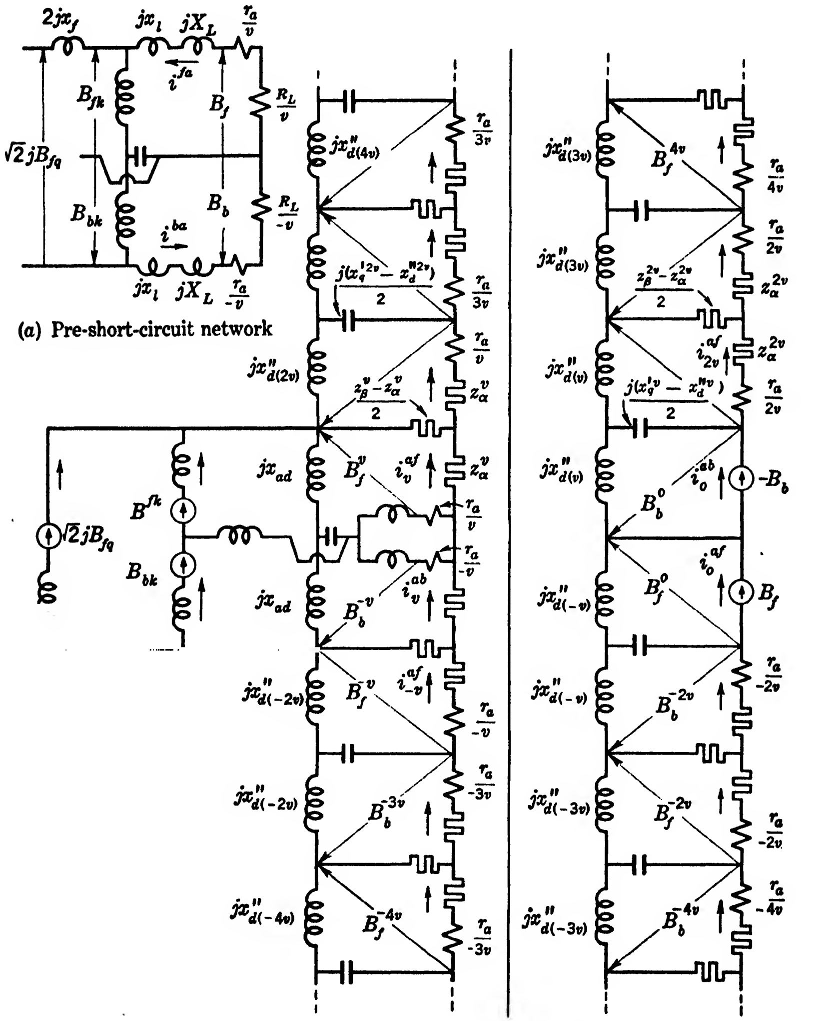
Hence the constant torque is the sum of twelve components, the fundamental-frequency ($v$) torques consist of sixteen components, and the double-frequency torques consist of eight components. The double-frequency torque, for instance, has the following components:

$$T_{2v} = i_{2v}{}^f B_b{}^0 + i_0{}^{f*} B_b{}^{-2v*} + i_{-2v}{}^{b*} B_f{}^{0*} + i_0{}^b B_f{}^{2v} + \cdots$$

$$+ i_{2v}{}^f B_f{}^{0*} + i_0{}^{f*} B_f{}^{2v} + i_{-2v}{}^{b*} B_b{}^0 + i_0{}^b B_b{}^{-2v*} \qquad 12.3$$

## THROWING AN UNBALANCED LOAD ON AN ALTERNATOR

When an *unbalanced* load is thrown on an open-circuited synchronous machine, or on one supplying some balanced load, the post-short-circuit networks contain time harmonics (Fig. 12.2). All networks are identical with those of Fig. 12.1, except that the two post-short-circuit networks are enlarged with the higher time-harmonic meshes, because of the presence of the unbalanced mutual impedances.



(a) Pre-short-circuit network

(b) Trapped field fluxes impressed (c) Trapped armature fluxes impressed

FIG. 12.2. Throwing an unbalanced load on an alternator.

The torques are calculated in the same manner as in the previous case, with the difference that in the original equations more flux and current components appear.

## SINGLE-PHASE SHORT CIRCUIT OF AN ALTERNATOR

When a line-to-line short circuit occurs on an alternator on open circuit, the networks of Fig. 12.3 are identical with those of Fig. 12.2,

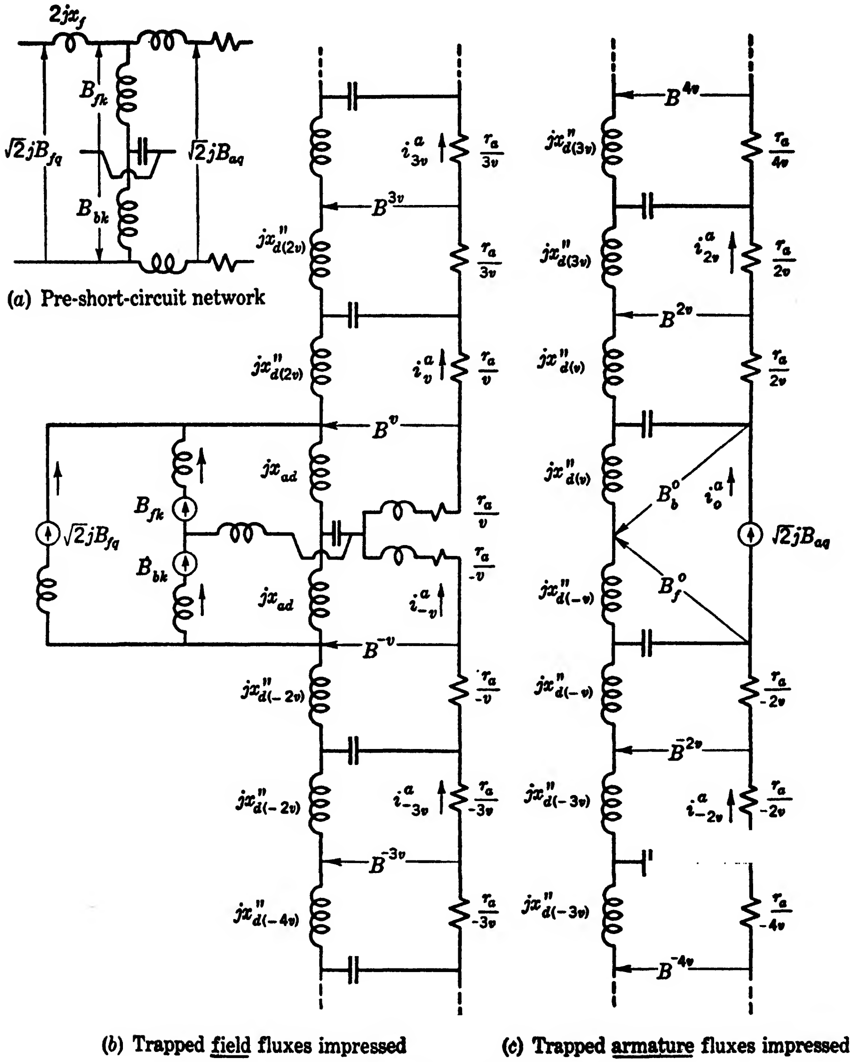



FIG. 12.3. Single-phase short circuit of an alternator.

except that the unbalanced load impedances are infinite and the corresponding branches are open.

The constant and harmonic torques are calculated again by Tables 12.1 and 12.2. Now only one type of current and flux exists.

## TURN-TO-TURN SHORT CIRCUIT OF A DOUBLE-WINDING GENERATOR

Let a double-winding generator supply a balanced load and let one of the armature *turns* be short-circuited. Since the shorted turn has

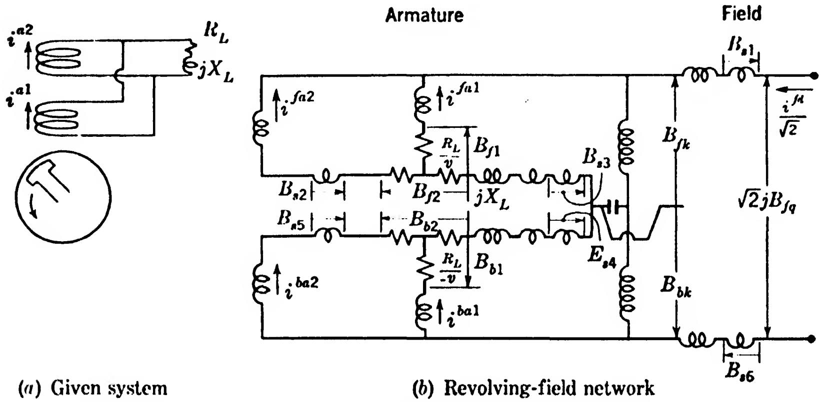



FIG. 12.4. Double-winding generator (pre-short-circuit network).

different mutual reactances with the two armature windings, both armature windings must be shown even in the pre-short-circuit network.

The equivalent circuit of a synchronous machine with *two armature windings* was shown in Fig. 6.6*a*. When two windings are in parallel

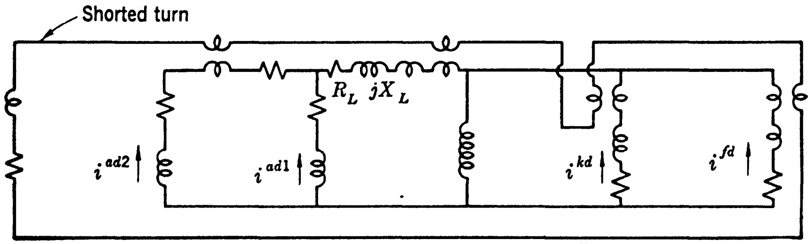
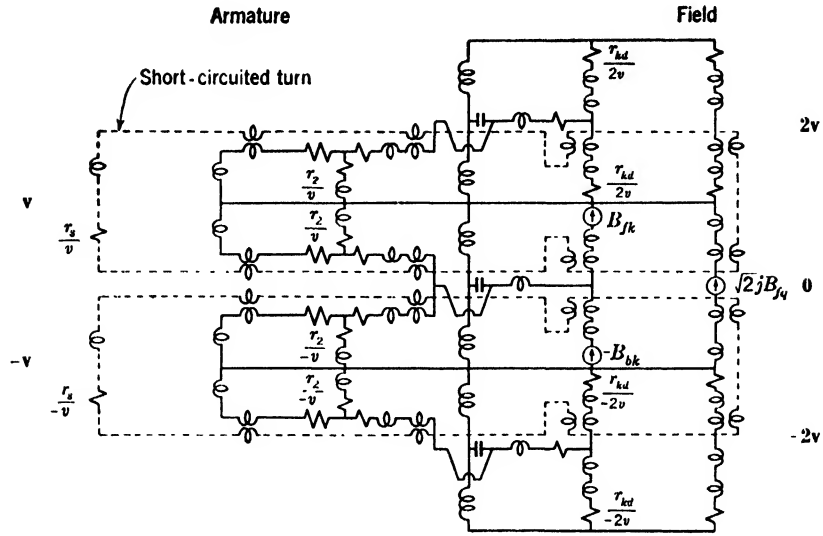
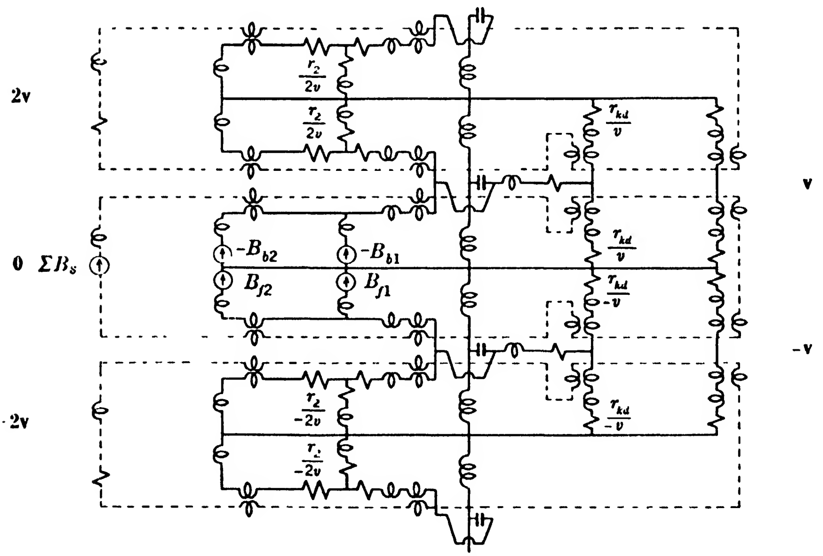



FIG. 12.5. Short-circuited turn (direct-axis network).

and are connected to a balance load, the resulting equivalent circuit is shown in Fig. 12.4. The load $Z_L$ is connected to that branch of the equivalent circuit in which the resultant current $i^1 + i^2$ flows. This network will also serve as the pre-short-circuit network.




(*a*) Trapped field fluxes impressed




(*b*) Trapped armature fluxes impressed

Fig. 12.6. Turn-to-turn short circuit of a double-winding generator.

*The shorted turn may be considered,* in an approximate manner, *as a third layer of single-phase armature winding,* which, however, has a mutual reactance not only with the armature windings but also with the field and amortisseurs. For that reason it must be represented as an *electrically isolated mesh.* Figure 12.5 shows it along the direct axis. This shorted coil will appear only in the post-short-circuit network. However, in the pre-short-circuit network the voltages induced in this shorted turn $(B_{si} + B_{s2} + \cdots)$ have to be indicated.

The post-short-circuit phenomenon for the three-winding generator of Fig. 12.6 is analogous to that of the one-winding generator of Fig. 12.2. Since the original two-winding generator is *balanced, the various time-harmonic networks are isolated.* They are coupled together only by the shorted turn (a single-phase armature winding) shown by dotted lines for better visualization. (Note how the single-phase mesh of the shortest turn is staggered with the single-phase mesh of the field.) Because of the mutual reactances, the field meshes cannot be so easily replaced by operational impedances.

If the shorted turn disappears, the remaining networks give the sudden three-phase short circuit of a two-winding generator, since the higher time-harmonic networks become isolated, with no voltages impressed on them.

## SELF-EXCITATION

It has been assumed that all trapped fluxes become d-c fluxes for the first few instants, when viewed from a physical reference frame rigidly connected to the conductors (or to the magnetic structures). As the machine rotates, these d-c fluxes remain constant, even though no voltages are impressed from the outside to maintain the fluxes constant.

Since no *active* power may enter the machine from the impressed d-c fluxes, the constant flux-linkage theorem is equivalent to the assumption that the rotating machine acts as a self-excited system, as far as the transients are concerned. In consequence, *the impedance of the machine and of its equivalent circuit, viewed from each d-c flux, is a pure reactance* $jX$. (The transient $i^2r$ losses of the *machine* are supplied by the rotor torque only. In the *equivalent circuit* the transient $i^2r$ losses are zero.)

It must be emphasized that the measured impedance of the equivalent circuit appears as a pure reactance only if the following conditions are satisfied:

1. The reference frame of the mesh, where the input impedance is measured (where $B$ is impressed), must be rigidly connected to the structure, because only in such a frame does $B$ appear as a d-c flux.

2. The reference frame must be a physical (**d** and **q**) and not a hypothetical (**f** and **b**) frame.

## THE DECREMENT FACTORS

The decrement factor of a flux trapped in any mesh is defined as $X/r$ where:

1. $X$ is the impedance of the network (always a pure reactance) viewed from the mesh, when all other meshes are *short-circuited.*
2. $r$ is the resistance of the mesh.

Again the decrement factor of a mesh is defined only if:

1. The mesh is expressed along the physical (**d** and **q**) frame.
2. The physical reference frame of the mesh is rigidly connected to the conductors.

When the decrement factor of a flux is known, the value of $B$ at any time after a short circuit is

$$B = B_0 \epsilon^{-(r/x)t}$$

## DETERMINATION OF THE NETWORK REACTANCE

When a winding is single-phase, so that in its equivalent circuit $i^d$ flows, its reactance $X$ is found simply by measuring the input impedance. In Fig. 12.3*c* *all positive and negative resistances are symmetrically arranged,* so that even by inspection the input impedance is a pure reactance.

When a complete layer of winding exists (as in the amortisseur or in the polyphase armature) the equivalent circuit gives, not the **d** and **q** reactances, but only the **f** and **b** reactances. Since the **d**, **q** reactances, $X$, are short-circuit reactances (all meshes being shorted), they *cannot* be found from the **f**, **b** short-circuit impedances by the sequence transformations of Eq. 2.4. However, the **d**, **q** short-circuit reactance may be found in a roundabout way from the **f**, **b** open-circuit reactances (which may be measured) as follows:

1. Let it be assumed that the **d**, **q**, open-circuit reactances are known as

$$\begin{aligned} e_d &= x_1 i^d + x_2 i^q \\ e_q &= x_3 i^d + x_4 i^q \end{aligned} \qquad 12.4$$

2. Then the needed short-circuit reactances can be expressed in terms of the open-circuit reactances by eliminating $i^q$ from the second equation (or $i^d$ from the first),

$$\begin{aligned} e_d &= i^d X_d{}^1 \quad (\text{if } e_q = 0) \\ e_q &= i^q X_q{}^1 \quad (\text{if } e_d = 0) \end{aligned} \qquad 12.5$$

where the short-circuit reactances are

$$X_d{}^1 = x_1 - x_2x_3/x_4 \qquad\qquad 12.6$$
$$X_q{}^1 = x_4 - x_3x_2/x_1$$

3. If the **d**, **q** open-circuit reactances (Eq. 12.3) are transformed into the **f**, **b** open-circuit reactances by Eq. 2.4, the latter become

$$2e_f = (a + jb)i^f + (c - jd)i^b \qquad\qquad 12.7$$
$$2e_b = (c + jd)i^f + (a - jb)i^b$$

where the new reactances are defined in terms of the old ones as

$$a = x_1 + x_4 \qquad\qquad c = x_1 - x_4 \qquad\qquad 12.8$$
$$b = x_3 - x_2 \qquad\qquad d = x_2 + x_3$$

These self-impedances and mutual impedances (not pure reactances) may be measured.

4. The needed **d**, **q** short-circuit reactances may now be expressed in terms of the measurable **f**, **b** open-circuit reactances as

$$X_d{}^1 = \frac{a^2 + b^2 - (c^2 + d^2)}{a - c} \qquad X_q{}^1 = \frac{a^2 + b^2 - (c^2 + d^2)}{a + c} \qquad 12.9$$

Hence the $X$ occurring in the decrement factors of the direct-axis and quadrature-axis windings of any layer are found by the following steps:

1. Open the **b** mesh.
2. Impress unit current in the **f** mesh.
3. The impressed voltage in the **f** mesh is $a + jb$.
4. The open-circuit voltage in the **b** mesh is $c + jd$.
5. Use Eq. 12.9 to find $X_d{}^1$ and $X_q{}^1$.

## AMORTISSEUR AND FIELD DECREMENT FACTORS

Since in a synchronous machine the field resistance is very small compared with the amortisseur resistances, it is customary to assume that:

1. In measuring the amortisseur decrements the field is short-circuited ($r_f = 0$).
2. In measuring the field decrement the amortisseur windings are open-circuited ($r_{kd} = r_{kq} = \infty$).

# EPILOGUE

## The Electrodynamics of Equivalent Circuits

## 1. THE DYNAMICAL EQUATIONS OF ROTATING ELECTRIC MACHINERY

### THE FOUR BASIC TYPES OF REFERENCE FRAMES

It was shown in connection with Table 4.1 that the reference frames of rotating electric machines may be grouped into four main classes according to the manner of attachment of the reference frames to the magnetic structures. (The magnetic and electric materials, namely the iron and copper, are always rigidly connected to each other in electric machinery.) The form that the dynamical equations of electric machines assumes is influenced by the type of reference frame introduced. The four basic sets of equations, corresponding to the four basic types of reference frames of Table 4.1, will now be presented.

### THE HOLONOMIC, RIEMANNIAN REFERENCE FRAME I

The most obvious assumption for the analysis of a rotating machine is that the reference axes of all stator windings should be rigidly connected to the stator iron and copper and that those of the rotor windings should be rigidly connected to the rotor iron and copper. In this frame the physics of the phenomenon is very simple and the transient voltage equation along each axis has the form

$$e = Ri + p\psi$$

This is the reference frame (although written for three phases instead of two) which is used by Park as a starting point in the development of his synchronous machine equations.* This is also the reference frame

* R. H. Park, "Two-Reaction Theory of Synchronous Machinery," *Transactions of the AIEE*, Vol. 48, pp. 716–727, July 1929.

along which books on physics and electrotechnics employ the "holonomic" form of the dynamical equations of Lagrange,

$$f_\alpha = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^\alpha}\right) - \frac{\partial T}{\partial x^\alpha} + \frac{\partial F}{\partial \dot{x}^\alpha} \tag{1}$$

accompanied by the statement that these equations represent the dynamical theory of "all" electric machines. This dynamical theory originally was developed by Maxwell almost a century ago, and until recently no progress had been made by electrical engineers to extend it to commutator machines or to other reference frames of a synchronous or induction machine. The single dynamical equation includes all the *voltage* equations and the *torque* equation.

In terms of tensors the voltage equations may assume two different forms, depending upon whether the variables are both $\mathbf{i}$ and $\boldsymbol{\psi}$, or $\mathbf{i}$ only. In particular:

(*a*) In terms of $\mathbf{i}$ only:

$$\mathbf{e} = \mathbf{Ri} + p(\mathbf{Li})$$

$$\mathbf{e} = \mathbf{Ri} + \mathbf{L}p\mathbf{i} + \frac{\partial \mathbf{L}}{\partial \theta} p\theta\mathbf{i} \tag{2}$$

(*b*) In terms of $\mathbf{i}$ and $\boldsymbol{\psi}$:

$$\mathbf{e} = \mathbf{Ri} + p\boldsymbol{\psi}$$

The torque equation is

$$f = I\frac{d^2\theta}{dt^2} - \frac{1}{2}\frac{\partial}{\partial\theta}(\mathbf{i}^*\mathbf{Li}) = I\frac{d^2\theta}{dt^2} - \frac{1}{2}\frac{\partial}{\partial\theta}(\mathbf{i}^*\boldsymbol{\psi})$$

The resultant tensor equation, which includes both voltage and torque equations, is written as

$$e_{\bar{\alpha}} = R_{\bar{\alpha}\beta}i^\beta + L_{\bar{\alpha}\beta}\frac{di^\beta}{dt} + [\bar{\beta}\gamma, \bar{\alpha}]i^{\bar{\alpha}}i^\beta \tag{3}$$

Here $[\bar{\beta}\gamma, \bar{\alpha}]$ is the "holonomic" form of the Christoffel symbol defined as

$$[\bar{\beta}\gamma, \bar{\alpha}] = \frac{1}{2}\left(\frac{\partial L_{\gamma\bar{\alpha}}}{\partial x^\beta} + \frac{\partial L_{\bar{\alpha}\beta}}{\partial x^\gamma} - \frac{\partial L_{\bar{\beta}\gamma}}{\partial x^{\bar{\alpha}}}\right) \tag{4}$$

where $L_{\bar{\alpha}\beta}$ is the "metric tensor" representing all the self- and mutual inductances and the rotor inertia. This tensorial equation can be derived from the dynamical equation of Lagrange (1) simply by replacing $T$ by $L_{\bar{\alpha}\beta}i^{\bar{\alpha}}i^\beta/2$. Expressed in another way, the above equation is an

"explicit" form of the Lagrangian equation itself. (The bars over the indices assume that the variables $i^{\bar{\alpha}}$ are *complex* functions; hence the symbols represent "spinors" rather than "tensors.")

Since the dynamical equation of Lagrange will soon prove helpless, when other reference frames are to be introduced, it is necessary to give a geometrical interpretation to the above two dynamical equations, Eqs. 1 and 3. Geometrically, the equations represent the motion of a point particle in an $n$-dimensional Riemannian space, expressed along a holonomic reference frame. The geometrical language will rescue dynamics from an apparent breakdown.

## THE NON-HOLONOMIC, RIEMANNIAN FRAME II(a)

Of course, electrical engineers are aware that the previous equations can be used only in exceptional cases because of the analytical complexity introduced by the cos $\theta$, sin $\theta$ terms in the metric tensor **L**. New, more practical sets of equations are found—in which these trigonometric functions no longer occur—by assuming that both stator and rotor reference frames are rigidly connected to the stator only, or to the rotor only. This theory is now called the "two-reaction theory" in synchronous machines and "cross-field theory" in induction machines.

It is very important to note *that now the voltage equations do not have the same form as in the previous frame but contain an additional speed term $p\theta_r$ as*

$$e = ri + p\psi_d - \psi_q p\theta_r$$

Also the electrical torque assumes a different form:

$$f = i^d\psi_q - i^q\psi_d$$

The appearance of an additional speed term is due to the fact that the new reference frame—no longer connected to the structure—is a so-called "non-holonomic" reference frame to which the usual "holonomic" equations, Eqs. 1 and 3, no longer apply. This fact was recognized by the author,* who showed that the above equations correspond exactly to the "non-holonomic" form of the dynamical equations of Lagrange, given in many advanced books on dynamics.†

$$f_\alpha = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}^\alpha}\right) - \frac{\partial T}{\partial x^\alpha} + \frac{\partial F}{\partial \dot{x}^\alpha} + \frac{\partial T}{\partial \dot{x}^\alpha}\left(\frac{\partial C_k{}^\delta}{\partial x^n} - \frac{\partial C_n{}^\delta}{\partial x^k}\right)C_\alpha{}^k C_\beta{}^n \dot{x}^\beta \qquad 5$$

* G. Kron, "Non-Riemannian Dynamics of Rotating Electrical Machinery," *Journal of Mathematics and Physics*, 1934, pp. 103–194.

† E. T. Whittaker, *Analytical Dynamics*, p. 43. Cambridge University Press, 4th Ed., 1937.

The transformation tensor $C_\alpha{}^k$ always appears in such equations and relates the non-holonomic frame to the holonomic one. The term containing it, namely

$$\Omega_{\alpha\beta}{}^\delta = \left(\frac{\partial C_k{}^\delta}{\partial x^n} - \frac{\partial C_n{}^\delta}{\partial x^k}\right) C_\alpha{}^k C_\beta{}^n \qquad 6$$

is the so-called "non-holonomic object," a non-tensor geometric object that will play an important part presently.

In terms of tensors the voltage and torque equations are (if $\boldsymbol{\gamma}$ is the "coefficient of rotation" containing only $\pm 1$):

(*a*) In terms of **i** only:

$$\mathbf{e} = \mathbf{Ri} + p(\mathbf{Li}) + \boldsymbol{\gamma}\mathbf{L}p\theta_r\mathbf{i}$$

(*b*) In terms of **i** and $\boldsymbol{\psi}$:

$$\mathbf{e} = \mathbf{Ri} + p\boldsymbol{\psi} + \boldsymbol{\gamma}\boldsymbol{\psi}p\theta_r \qquad 7$$

The torque equation is

$$f = I\frac{d^2\theta}{dt^2} - \frac{1}{2}\frac{\partial}{\partial\theta_t}(\mathbf{i}^*\mathbf{Li}) - \mathbf{i}^*(\boldsymbol{\gamma}\mathbf{L})\mathbf{i}$$

$$= I\frac{dv}{dt} - \frac{1}{2}\frac{\partial}{\partial\theta_t}(\mathbf{i}^*\boldsymbol{\psi}) - \mathbf{i}^*\boldsymbol{\gamma}\boldsymbol{\psi}$$

In these equations $\boldsymbol{\gamma}$ is a tensor.

It should be noted that $p\psi = p(\mathbf{Li})$ is the sum of two terms.

$$p\boldsymbol{\psi} = p(\mathbf{Li}) = \mathbf{L}p\mathbf{i} + \frac{\partial\mathbf{L}}{\partial\theta_t}p\theta_t\mathbf{i}$$

where $p\theta_t = v_t$ represents the speeds of the *time-harmonic* reference frames rotating with $2nv_r$ with respect to the rotor (Chapter 11). This last term is part of the Christoffel symbol (Eq. 4).

The resultant tensorial equation is

$$e_{\bar\alpha} = R_{\bar\alpha\beta}i^\beta + L_{\bar\alpha\beta}\frac{di^\beta}{dt} + [\bar\beta\gamma, \bar\alpha]i^{\bar\beta}i^\gamma \qquad 8$$

where $[\bar\beta\gamma, \bar\alpha]$ is now the "non-holonomic" form of the Christoffel symbol, containing the non-holonomic object $\Omega_{\bar\beta\gamma,\bar\alpha}$ of Eq. 6 also in its definition, in addition to Eq. 4. Geometrically, both the Lagrangian and tensorial equations, Eqs. 5 and 8, may still be pictured as the motion of a point in an $n$-dimensional Riemannian space, which, however, now is rigged

up at every point by a non-holonomic reference frame, instead of a holonomic one.

## THE HOLONOMIC, NON-RIEMANNIAN FRAME II(*b*)

It was shown by the author (in the last reference) that because of the "cylindrical" properties of the variables (for instance, the relations between the currents, namely $C_\alpha^{\alpha'}$, are functions only of $\theta$) *there is nothing in the new equations to indicate that they were derived from some holonomic frame.* For that reason *it is possible to consider them, on their own strength, as basically new equations,* having their own special physical interpretation. Such a reformulation is accomplished by considering the expression in Eq. 6, defining the non-holonomic object $\Omega_{\alpha\beta,\gamma}$, not a non-tensor object, but a tensor, the so-called "torsion tensor" $S_{\bar\beta\gamma\bar\alpha}$. In consequence $\boldsymbol{\gamma}$ in the equation of voltage, Eq. 7, also becomes a tensor, the "rotation tensor."

Although this reformulation becomes mathematically useful only in going over to a new reference frame (as $S_{\bar\beta\gamma\bar\alpha}$ has a different law of transformation from that of $\Omega_{\beta\gamma,\alpha}$), it becomes physically important even in this particular reference frame. Since $\boldsymbol{\psi}$ is always a tensor and $\boldsymbol{\gamma}$ is now a tensor (because of the redefinition), their product $\boldsymbol{\gamma\psi}$ must necessarily also be a tensor, *a new physical entity,* the so-called "flux-density" **B** of the armature. This is the same physical concept of **B** that occurs in Maxwell's field equations. (The flux linkage $\boldsymbol{\psi}$ that occurred hitherto corresponds to the vector potential **A** of Maxwell.) The new tensorial voltage and torque equations become

(*a*) In terms of **i** only:

$$\mathbf{e} = \mathbf{Ri} + p(\mathbf{Li}) + \mathbf{Gi}p\theta_r$$

(*b*) In terms of **i** and $\boldsymbol{\psi}$:

$$\mathbf{e} = \mathbf{Ri} + p\boldsymbol{\psi} + \boldsymbol{\gamma\psi}p\theta_r$$

(*c*) In terms of **i**, $\boldsymbol{\psi}$, and **B**:

$$\mathbf{e} = \mathbf{Ri} + p\boldsymbol{\psi} + \mathbf{B}p\theta_r$$

9

The torque equation is

$$f = I\frac{d^2\theta_r}{dt^2} - \frac{1}{2}\frac{\partial}{\partial\theta_t}(\mathbf{i^*Li}) - \mathbf{i^*Gi}$$

$$= I\,\frac{dv}{dt} - \frac{1}{2}\frac{\partial}{\partial\theta}(\mathbf{i}^*\boldsymbol{\psi}) - \mathbf{i^*B}$$

In these equations $\boldsymbol{\gamma}$ is a tensor.

This physical reinterpretation has several advantages. Among others:

1. It enables us to assume that the new equations are also valid for *commutator* machines. In such machines it is an absolute necessity to introduce both physical concepts $\boldsymbol{\psi}$ and **B**, as there exist no simple relations between "flux linkage" $\boldsymbol{\psi}$ and "flux density" **B**, and these concepts must be and are considered independent of each other. (That is, in commutator machines **B** is expressible as $\boldsymbol{\gamma\psi}$ only in special cases. See Chapter 7.)

2. The torque can now be expressed as a generalization of the force equation of Maxwell, namely of $f = iB$ (for the fundamental wave only).

3. The concept of **B** enables the construction of equivalent circuits on a more logical basis. In assuming symmetrical components, it is possible to measure on the equivalent circuit the components of **B** as differences of potential.

It is emphasized that all three forms of the equations (containing **i** only, or **i** and $\boldsymbol{\psi}$, or **i**, $\boldsymbol{\psi}$, and **B**) are equivalent mathematically and the choice among them depends on convenience only. One form may sometimes be shorter; another may give a more visualizable physical picture. For instance, if the field axes of a synchronous machine are permanently eliminated by the use of operational impedances, then the use of **i** and $\boldsymbol{\psi}$ requires the least number of symbols.

Unfortunately, conventional rigid dynamics has no equations—analogous to the Lagrangian forms, Eqs. 1 and 5—that could represent this new dynamical system containing **B** or, for that matter, that could represent a d-c machine with commutators. Geometry comes now to the rescue. The resultant tensorial equation (including both voltage and torque equations) can be written in the form of a "shortest" line between two points:

$$e_{\bar{\alpha}} = R_{\bar{\alpha}\beta}i^{\beta} + L_{\bar{\alpha}\beta}\frac{di^{\beta}}{dt} + \Gamma_{\bar{\beta}\gamma,\bar{\alpha}}i^{\bar{\beta}}i^{\gamma}$$

$$e_{\bar{\alpha}} = R_{\bar{\alpha}\beta}i^{\beta} + L_{\bar{\alpha}\beta}\frac{di^{\beta}}{dt} + [\bar{\beta}\gamma, \bar{\alpha}]i^{\bar{\beta}}i^{\gamma} - 2S_{\bar{\alpha}\bar{\beta}\gamma}i^{\bar{\beta}}i^{\gamma} \qquad 10$$

This equation is also called, sometimes, the generalized Lagrangian equation, as it is derivable from a variational principle. Here $\Gamma_{\bar{\beta}\gamma,\bar{\alpha}}$ is called the "affine connection" and contains, besides the holonomic Christoffel symbol $[\bar{\beta}\gamma, \bar{\alpha}]$, the torsion tensor $S_{\bar{\beta}\gamma,\bar{\alpha}}$ in its definition. (If time harmonics are also considered, $[\bar{\beta}\gamma, \bar{\alpha}]$ is not zero.)

Geometrically the new equation represents the motion of a point in a non-Riemannian space rigged up with a holonomic frame (instead of a Riemannian space with non-holonomic frames). That the frame still

may be considered holonomic is seen from the fact that all the reference axes are still connected rigidly to *one* of the magnetic structures. The appearance of the non-Riemannian space (a space with "torsion") is due to the motion of the conductors with respect to the reference axes. That is, even though the observer sees the current-density wave stationary, the conductors themselves replace one another in succession (the "metallic metabolism" of Hoffmann*).

## THE "ELECTROMAGNETIC-FIELD TENSOR" $F_{\bar{a}\beta}$

The appearance of a new physical entity, due to the reinterpretation, is shown more emphatically by the introduction in the last tensor equation of a skew-symmetric "electromagnetic-field" tensor $F_{\bar{a}\beta}$ containing **B** (for the voltage) and −**B** (for the torque). In terms of this important tensor, the equation of motion of a rotating machine, Eq. 10, becomes

$$e_{\bar{a}} = R_{\bar{a}\beta}i^{\beta} + L_{\bar{a}\beta}\frac{di^{\beta}}{dt} + [\bar{\beta}\gamma, \bar{\alpha}]i^{\beta}i^{\gamma} + F_{\bar{a}\beta}i^{\beta} \tag{11}$$

where $[\bar{\beta}\gamma, \bar{\alpha}]$ is the holonomic Christoffel symbol. *This equation is exactly the same as Eq. 3, except for the presence of the added term* $F_{\bar{a}\beta}i^{\beta}$. *Hence a physical entity is made to appear or disappear at will by a mere change in the reference frames.*

If $e_{\bar{a}}$ and $R_{\bar{a}\beta}$ are made equal to zero, this last equation also represents the classical or relativistic motion of a charged particle in an electromagnetic field. This tensorial equation of motion also plays a basic role in the theories of elementary particles of nuclear physics.

## THE NON-HOLONOMIC, NON-RIEMANNIAN FRAMES III AND IV

In interconnecting several machines (Chapter 9), some or all of the reference frames are no longer connected rigidly to the magnetic structure but are free of them. Separating the reference frame from the magnetic structure is again equivalent to the introduction of a non-holonomic reference frame. In analogy to Eq. 7 an additional speed term $p\theta_f$ appears (in addition to the rotor speed $p\theta_r$), so that a representative circuit equation has the form

$$e = Ri + p\psi + Bp\theta_r + \phi p\theta_f$$

where $\phi$ is a new type of flux density produced by the currents passing through the freely rotating reference frame.

* B. Hoffmann, "Kron's Non-Riemannian Electrodynamics," Einstein's 70th birthday commemorative issue of the *Reviews of Modern Physics*, Vol. 21, No. 3, pp. 535–540, July 1949.

The tensorial voltage equation assumes now four different forms:

$$
\begin{aligned}
&(a) \qquad \mathbf{e} = \mathbf{Ri} + p(\mathbf{Li}) + \mathbf{Gi}p\theta_r + \mathbf{Vi}p\theta_f \\
&(b) \qquad \mathbf{e} = \mathbf{Ri} + p\boldsymbol{\psi} + \boldsymbol{\gamma}_1\boldsymbol{\psi}p\theta_r + \boldsymbol{\gamma}_2\boldsymbol{\psi}p\theta_f \\
&(c) \qquad \mathbf{e} = \mathbf{Ri} + p\boldsymbol{\psi} + \mathbf{B}p\theta_r + \boldsymbol{\gamma}_2\boldsymbol{\psi}p\theta_f \\
&(d) \qquad \mathbf{e} = \mathbf{Ri} + p\boldsymbol{\psi} + \mathbf{B}p\theta_r + \boldsymbol{\phi}p\theta_f
\end{aligned}
\tag{12}
$$

Along the new, uniformly rotating reference frame the torque equation remains unchanged, and the resultant tensorial equation, representing also the generalized Lagrangian equation, assumes the form

$$
\begin{aligned}
e_{\bar{\alpha}} &= R_{\bar{\alpha}\beta}i^{\beta} + L_{\bar{\alpha}\beta}\frac{di^{\beta}}{dt} + \Gamma_{\bar{\beta}\gamma,\bar{\alpha}}i^{\bar{\beta}}i^{\gamma} \\
e_{\bar{\alpha}} &= R_{\bar{\alpha}\beta}i^{\beta} + L_{\bar{\alpha}\beta}\frac{di^{\beta}}{dt} + \left(\frac{\partial L_{\gamma\bar{\alpha}}}{\partial x^{\bar{\beta}}} - \frac{1}{2}\frac{\partial L_{\bar{\beta}\gamma}}{\partial x^{\bar{\alpha}}}\right)i^{\bar{\beta}}i^{\gamma} - 2\Omega_{\bar{\alpha}\bar{\beta},\gamma}i^{\bar{\beta}}i^{\gamma} - 2S_{\bar{\alpha}\bar{\beta}\gamma}i^{\bar{\beta}}i^{\gamma}
\end{aligned}
\tag{13}
$$

where the effine connection $\Gamma_{\bar{\beta}\gamma,\bar{\alpha}}$ contains now the non-holonomic form of $[\bar{\beta}\gamma, \bar{\alpha}]$—hence the non-holonomic object $\Omega_{\bar{\beta}\gamma,\bar{\alpha}}$—in addition to the torsion tensor $S_{\bar{\beta}\gamma\bar{\alpha}}$. Geometrically the equations describe the motion of a point in a non-Riemannian space, rigged up with a non-holonomic reference frame.

# 2. REDUCTION OF THE DYNAMICAL EQUATIONS TO EQUIVALENT CIRCUITS

## THE ABSOLUTE TIME DERIVATIVE

The most general form of the dynamical equation of a rotating machine, Eq. 13,

$$e_{\bar{\alpha}} = R_{\bar{\alpha}\beta} i^{\beta} + L_{\bar{\alpha}\beta} \frac{di^{\beta}}{dt} + \Gamma_{\bar{\beta}\gamma,\bar{\alpha}} i^{\bar{\beta}} i^{\gamma}$$

may be expressed in a form in which each symbol is a tensor:

$$e_{\bar{\alpha}} = R_{\bar{\alpha}\beta} i^{\beta} + L_{\bar{\alpha}\beta} \frac{\delta i^{\beta}}{dt} \qquad 14$$

The "absolute" (or "covariant") derivative is defined as

$$\frac{\delta i^{\delta}}{dt} = \frac{di^{\delta}}{dt} + \Gamma_{\bar{\beta}\gamma}{}^{\delta} i^{\bar{\beta}} i^{\gamma} \qquad 15$$

Since the absolute time derivative of the metric tensor is zero in electric machinery,

$$\frac{\delta L_{\bar{\alpha}\beta}}{dt} = 0 \qquad 16$$

the equation of motion may also be written as

$$e_{\bar{\alpha}} = R_{\bar{\alpha}\beta} i^{\beta} + \frac{\delta}{dt}\left(L_{\bar{\alpha}\beta} i^{\beta}\right) \qquad 17$$

## RESTRICTIONS ON THE EQUATIONS

Let it be assumed that the equations apply to only *one* machine. Then the equations contain only tensors of valence one and two. Using the

*direct* notation, the equation of motion (Eq. 13) splits into the equation of voltage and torque.

$$\begin{aligned} \mathbf{e} &= \mathbf{Ri} + \mathbf{L}\frac{d\mathbf{i}}{dt} + \frac{\partial \mathbf{L}}{\partial \theta_t} p\theta_t \mathbf{i} + \mathbf{G} p\theta_r \mathbf{i} + \mathbf{V} p\theta_f \mathbf{i} \\ T &= I\frac{dv_r}{dt} - \frac{1}{2}\mathbf{i}^* \frac{\partial \mathbf{L}}{\partial \theta_t}\mathbf{i} - \mathbf{i}^*\mathbf{G}\mathbf{i} \end{aligned} \qquad 18$$

(The torque now is denoted by $T$ instead of $f$ to avoid confusion with frequency $f$.)

*Equivalent circuits have been established in this book for those rotating machines only, for which the torsion tensor* **G** (a component of $S_{\alpha\beta\gamma}$) *may be expressed in terms of the metric tensor* **L** (see Chapter 7). In all induction and synchronous machines and in several types of a-c polyphase commutator machines **G** may be expressed as

$$\boldsymbol{\gamma}_r \mathbf{L} \qquad 19$$

where $\boldsymbol{\gamma}_r$ is the rotation tensor.

Along the physical **d**, **q** axes the rotation tensor has the *skew-symmetric* form

$$\boldsymbol{\gamma}_r = \begin{array}{c|cc} & \mathbf{d} & \mathbf{q} \\ \hline \mathbf{d} & & -1 \\ \mathbf{q} & 1 & \end{array} \qquad 20$$

Along the sequence axes it assumes a diagonal form:

$$\boldsymbol{\gamma}_r = \begin{array}{c|cc} & \mathbf{f} & \mathbf{b} \\ \hline \mathbf{f} & j & \\ \mathbf{b} & & -j \end{array} \qquad 21$$

In some a-c commutator machines the $j - s$ may be multiplied by a scalar $k$ (Chapter 7) to take care approximately of the difference between **G** and **L**.

$$\boldsymbol{\gamma}_r = \begin{array}{c|cc} & \mathbf{f} & \mathbf{b} \\ \hline \mathbf{f} & jk & \\ \mathbf{b} & & -jk \end{array} \qquad 22$$

## RESTRICTION ON THE REFERENCE FRAMES

Moreover, *equivalent circuits have been established in this book only for those reference frames in which the non-holonomic object* **V** (a special case of $\Omega_{\alpha\beta,\gamma}$) *and the Christoffel symbol* $\partial\mathbf{L}/\partial\theta_t$ *may be expressed in terms of the metric tensor* **L** *in the following manner:*

$$\mathbf{V} = \boldsymbol{\gamma}_f\mathbf{L} \qquad \text{and} \qquad \frac{\partial\mathbf{L}}{\partial\theta_t} = \boldsymbol{\gamma}_t\mathbf{L} \tag{23}$$

where $\boldsymbol{\gamma}_f$ and $\boldsymbol{\gamma}_t$ assume the same form as $\boldsymbol{\gamma}_r$. In the presence of time harmonics and space harmonics, $\boldsymbol{\gamma}_f$ and $\boldsymbol{\gamma}_t$ are multiplied by a scalar $k$ as in Eq. 22.

Hence the types of rotating machines and the types of reference frames for which equivalent circuits have been established both are slightly restricted by conditions expressed in Eqs. 19 and 23. For the more general types of machines and reference frames more general types of equivalent circuits have to be developed. For instance, this book does not develop any equivalent circuit even for a machine that has salient poles on both its stator and rotor.

## THE TWO COMPONENTS OF THE ABSOLUTE TIME DERIVATIVE

When such machines and such reference frames are assumed for which the above restrictions apply, the equation of voltage and torque (Eq. 18) may be written as

$$\begin{aligned} \mathbf{e} &= \mathbf{Ri} + (\mathbf{I}p + \boldsymbol{\gamma}_r p\theta_r + \boldsymbol{\gamma}_f p\theta_f + \boldsymbol{\gamma}_t p\theta_t)\mathbf{Li} \\ T &= I\frac{dv_r}{dt} - \mathbf{i}^*(\boldsymbol{\gamma}_r + \boldsymbol{\gamma}_t)\mathbf{Li} \end{aligned} \tag{24}$$

where **I** is the unit tensor. The expression in the first parentheses is the component of the original absolute time derivative (Eq. 17) along the *electrical* indices only,

$$\mathbf{e} = \mathbf{Ri} + \frac{\delta}{dt}(\mathbf{Li}) = \mathbf{Ri} + \frac{\delta}{dt}\boldsymbol{\phi} \tag{25}$$

and it may be written as a tensor of valence two in the form of a matrix:

$$\frac{\delta}{dt} = \frac{d}{dt}\mathbf{I} + \boldsymbol{\gamma}_r p\theta_r + \boldsymbol{\gamma}_f p\theta_f + \boldsymbol{\gamma}_t p\theta_t \tag{26}$$

The torque equation also may be written

$$T = I\frac{\delta}{dt}(v_r) \tag{27}$$

where $\delta/dt$ is the component of the original absolute time derivative (Eq. 17) along the *geometrical* index. Although this component of $\delta/dt$ is a scalar, *it cannot be factored out* in the form of Eq. 26.

## THE ABSOLUTE FREQUENCY TENSOR

If sinusoidal quantities are introduced, then

$$p = fj\omega, \qquad p\theta = v\omega$$

and the absolute time derivative along the electrical axes becomes

$$\frac{\delta}{dt} = j\omega[f\mathbf{i} - j\boldsymbol{\gamma}_r v_r - j\boldsymbol{\gamma}_f v_f - j\boldsymbol{\gamma}_t v_t] \tag{28}$$

*Since along the sequence axes each* $\boldsymbol{\gamma}$ *is a diagonal matrix and it is imaginary* (*Eq. 22*), *the expression in parentheses is a diagonal matrix containing only real numbers. It is the "absolute-frequency" tensor*

$$\mathbf{n} = f\mathbf{I} - j\boldsymbol{\gamma}_r v_r - j\boldsymbol{\gamma}_f v_f - j\boldsymbol{\gamma}_t v_t \tag{29}$$

The absolute time derivative of the flux linkages $\boldsymbol{\phi}$ becomes thereby in the presence of sinusoidal quantities

$$\frac{\delta}{dt} = j\omega\mathbf{n} \tag{30}$$

where $\mathbf{n}$ is a tensor with a diagonal real matrix along the sequence axes.

## REDUCTION TO OHM'S LAW

The equations of voltage becomes

$$\begin{aligned} \mathbf{e} &= \mathbf{Ri} + \frac{\delta}{dt}(\mathbf{Li}) \\ &= \mathbf{Ri} + \mathbf{n}j\omega\mathbf{Li} \end{aligned} \tag{31}$$

The *electrical* torque becomes, if the $j$ in $\boldsymbol{\gamma}_r$ and $\boldsymbol{\gamma}_t$ is factored out,

$$T_e = \mathbf{i}^* j\omega\mathbf{L}_r\mathbf{i}$$

where $\mathbf{L}_r\mathbf{i}$ represents only the *rotor* flux linkages.

If the reactance tensor is defined as

$$\mathbf{X} = \omega\mathbf{L}$$

the equations become

$$\begin{aligned} \mathbf{e} &= \mathbf{Ri} + \mathbf{n}j\mathbf{Xi} \\ &= (\mathbf{R} + \mathbf{n}j\mathbf{X})\mathbf{i} \\ T_e &= \mathbf{i}^* j\mathbf{X}_r\mathbf{i} \end{aligned} \tag{32}$$

The reactance tensor $j\mathbf{X}$ represents the magnetic field of the rotating machine at standstill. Since $\mathbf{L}$ is a symmetrical matrix with *positive* coefficients along the physical axes, $j\mathbf{X}$ *is symmetrical and can always be represented by at least an n-winding transformer* along the physical axes.

Multiplying the equation by $\mathbf{n}^{-1}$ *the stationary equivalent circuit of any rotating machine has the equations*

$$\begin{aligned} \mathbf{n}^{-1}\mathbf{e} &= (\mathbf{n}^{-1}\mathbf{R} + j\mathbf{X})\mathbf{i} = \mathbf{Zi} \\ T_e &= \mathbf{i}^* j\mathbf{X}_r\mathbf{i} = \mathbf{i}^*\mathbf{B} \end{aligned} \tag{33}$$

where $j\mathbf{X}_r\mathbf{i} = \mathbf{B}$ is a set of differences of potential measured across the inductors in the *rotor* meshes only.

It should be noted that *the resistances* $\mathbf{R}$ *and impressed voltages* $\mathbf{e}$ *are divided by the absolute-frequency tensor* $\mathbf{n}$, *just as was done in all equivalent circuits of this book.*

## RESTRICTIONS ON THE MAGNETIC FIELDS

In the impedance matrix

$$\mathbf{Z} = \mathbf{n}^{-1}\mathbf{R} + j\mathbf{X} \tag{34}$$

$j\mathbf{X}$ is always symmetrical along the physical axes and so is $\mathbf{R}$. However, *along the sequence axes,* $j\mathbf{X}$ *is not necessarily symmetrical, nor is* $\mathbf{n}^{-1}\mathbf{R}$. In the shaded-pole motor (Fig. 5.20) a phase shifter had to be introduced to take care of the non-symmetrical $j\mathbf{X}$ due to the presence of mutual impedance between the direct and quadrature axes.

## RESTRICTIONS ON THE LOADS

Since $\mathbf{R}$ represents the winding resistances, it also stands for any *outside* load-resistance also. In many problems the loads had to be restricted to *balanced* loads to make $\mathbf{n}^{-1}\mathbf{R}$ symmetrical. (See, for instance, the case of two unbalanced machines running at different speeds, Chapter 11. The load between them had to be assumed balanced.)

# APPENDIX 1

# REESTABLISHMENT OF THE TRANSIENT DYNAMICAL EQUATIONS FROM THE EQUIVALENT CIRCUITS

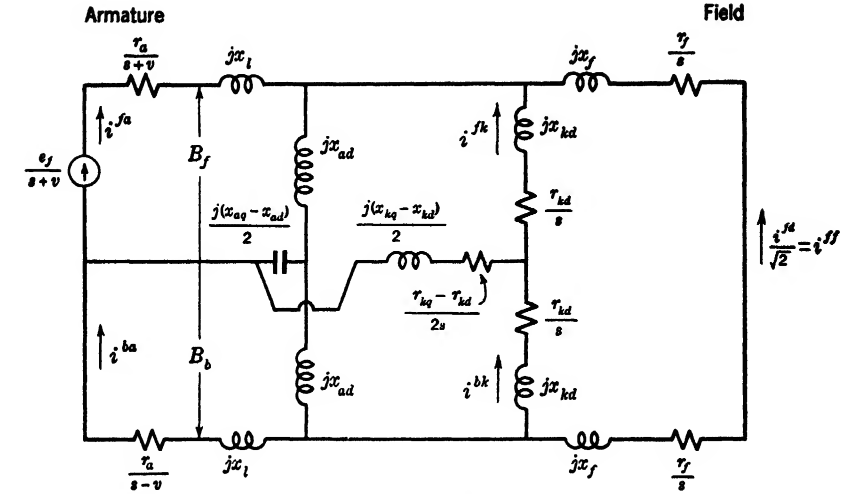
## ESTABLISHMENT OF THE STEADY-STATE EQUATIONS

It has been advocated by the author in his previous publications that, in order to establish the steady-state equations of performance of any machine (in some particular reference frame), it is preferable as a first step to establish the *transient* equations (in the same reference frame). In doing so no confusion may arise between the space phases and the time phases. The step from the transient to the steady-state equations is usually self-evident.

However, when the steady-state equivalent circuit of a machine and especially of a group of machines is available (in some particular reference frame), it is, of course, far more sensible to use the equivalent circuit itself as the starting point to write the steady-equations of a machine or a group of machines. Afterwards the found equations may be transformed to the desired reference frame.

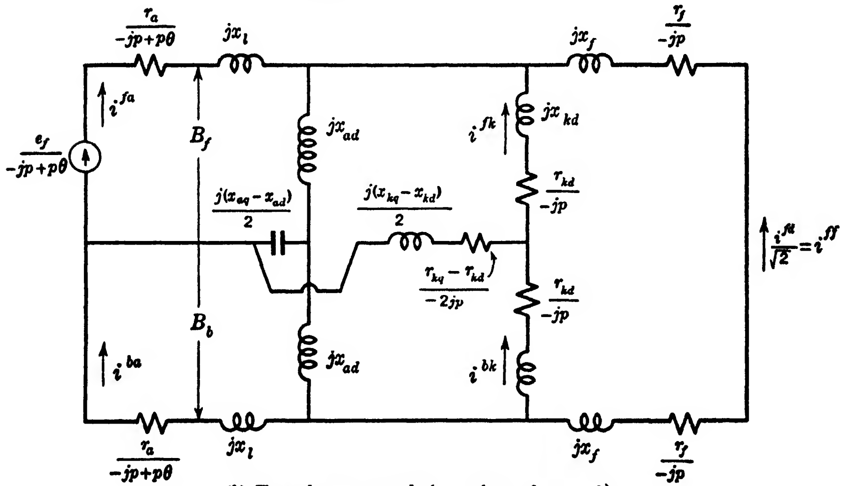
## ESTABLISHMENT OF THE TRANSIENT EQUATIONS

It is now suggested by the author that, if the transient equations of a machine and especially of a group of machines is desired, it is often preferable as a first step to establish for the group its *steady-state* equivalent circuit, containing the variable-frequency $f$ feature. The transient equation of the system is found, then, simply by replacing $f$ by $p/j = (d/dt)/j$. The transformation from the reference frame of the equiv-




(*a*) Variable frequency network ($f = s$) (Fig. 6.7*a*)

Torque $= i^{fa*}B_f + i^{ba*}B_b$




(*b*) Transient network ($s = -jp$ and $v = p\theta$)

FIG. A.1. *Transient* equivalent circuit of a salient-pole synchronous machine running below synchronous speed (sequence axes).

alent circuit to any other desired reference frame involves routine steps only.

Again attention is called to the importance of the variable-frequency characteristics of all equivalent circuits developed in this volume. Conventional equivalent circuits, even those of the polyphase induction motor, do not incorporate this feature; hence they cannot be used as models for writing transient equations of performance.

## SELF-IMPEDANCES OF A SYNCHRONOUS MACHINE

As an example of the method given, at the end of Chapter 4, to establish the *steady-state* and *transient* equations of any machine from its equivalent circuit, let the revolving-field equivalent circuit of a salient-pole synchronous machine running below synchronous speed (Fig. 6.7*A*) be considered (shown again in Fig. A.1*a*). It is possible to write five voltage equations $\mathbf{e} = \mathbf{Zi}$ for the five meshes, each mesh passing through the saliency reactance $(x_{aq} - x_{ad})/2$.

The network contains only airgap reactances $x_{ad}$, $x_{aq}$ and leakage reactances $x_l$, $x_{kd}$, $x_{kq}$, and $x_f$. In writing the equations the following self-impedances are defined:

$$x_d = x_{ad} + x_l \qquad X_{kd} = x_{ad} + x_{kd} \qquad X_f = x_{ad} + x_f$$

$$x_q = x_{aq} + x_l \qquad X_{kq} = x_{aq} + x_{kq}$$

These self-impedances (also the resistances), however, occur along the sequence axes, not isolated, but either as a *sum* term or as a *difference* term:

$$r_{kS} = \frac{r_{kd} + r_{kq}}{2} \qquad X_{kS} = \frac{X_{kd} + X_{kq}}{2}$$

$$r_{kD} = \frac{r_{kd} - r_{kq}}{2} \qquad X_{kD} = \frac{X_{kd} - X_{kq}}{2}$$

$$X_S = \frac{x_d + x_q}{2} \qquad X_{aS} = \frac{x_{ad} + x_{aq}}{2}$$

$$X_D = \frac{x_d - x_q}{2} \qquad X_{aD} = \frac{x_{ad} - x_{aq}}{2}$$

## EQUATIONS OF THE EQUIVALENT CIRCUIT

The voltage equations for the five meshes are (from Fig. A.1*a*) $\mathbf{e}_1 = \mathbf{Z}_1\mathbf{i}_1$, where

$$
\mathbf{Z}_1 = \begin{array}{c|c|c|c|c|c|}
 & \mathbf{f}_f & \mathbf{f}_k & \mathbf{b}_k & \mathbf{f}_a & \mathbf{b}_a \\
\hline
\mathbf{f}_f & 2\left(\frac{r_f}{s} + jX_f\right) & jx_{ad} & jx_{ad} & jx_{ad} & jx_{ad} \\
\hline
\mathbf{f}_k & jx_{ad} & \frac{r_{kS}}{s} + jX_{kS} & \frac{r_{kD}}{s} + jX_{kD} & jX_{aS} & jX_{aD} \\
\hline
\mathbf{b}_k & jx_{ad} & \frac{r_{kD}}{s} + jX_{kD} & \frac{r_{kS}}{s} + jX_{kS} & jX_{aD} & jX_{aS} \\
\hline
\mathbf{f}_a & jx_{ad} & jX_{aS} & jX_{aD} & \frac{r_a}{s+v} + jX_S & jX_D \\
\hline
\mathbf{b}_a & jx_{ad} & jX_{aD} & jX_{aS} & jX_D & \frac{r_a}{s-v} + jX_S \\
\end{array}
$$

In Fig. 6.7$a$ the slip $s$ has been replaced by the original base frequency $f$.

$$
\mathbf{i}_1 = \begin{array}{|c|c|c|c|c|}
\mathbf{f}_f & \mathbf{f}_k & \mathbf{b}_k & \mathbf{f}_a & \mathbf{b}_a \\
\hline
i^{ff} & i^{fk} & i^{bk} & i^{fa} & i^{ba} \\
\hline
\end{array}
$$

$$
\mathbf{e}_1 = \begin{array}{|c|c|c|c|c|}
\mathbf{f}_f & \mathbf{f}_k & \mathbf{b}_k & \mathbf{f}_a & \mathbf{b}_a \\
\hline
 & & & \frac{e_{fa}}{s+v} & \\
\hline
\end{array}
$$

It should be noted that the $\mathbf{Z}_1$ matrix is symmetrical. The differences of potential $B_f$ and $B_b$ are found as products of reactances and currents. The reactances may be arranged in the form of a matrix representing the torque tensor $\mathbf{G}_1$ so that the differences of potential $\mathbf{B}$ are expressed as $\mathbf{G}_1\mathbf{i}_1$. The torque tensor is

$$
\mathbf{G}_1 = \begin{array}{c|cc|c|c|c|}
 & \mathbf{f}_f & \mathbf{f}_k & \mathbf{b}_k & \mathbf{f}_a & \mathbf{b}_a \\
\mathbf{f}_a & jx_{ad} & jX_{aS} & jX_{aD} & jX_S & jX_D \\
\hline
\mathbf{b}_a & -jx_{ad} & -jX_{aD} & -jX_{aS} & -jX_D & -jX_S \\
\end{array}
$$

The torque is given by $T = \mathbf{i}_1{}^*\mathbf{G}_1\mathbf{i}_1$.

## THE STEADY-STATE SEQUENCE EQUATIONS

Let each equation be multiplied by the absolute frequency $n$, shown by its resistance. That is, let the first three rows of $\mathbf{Z}$ be multiplied by $s$, the fourth row by $s + v$, and the fifth row by $s - v$. Similarly, let the components of $\mathbf{e}$ be divided in the same manner. The absolute-frequency tensor is defined thereby as

$$\mathbf{n} = \begin{array}{c|ccccc} & \mathbf{f}_f & \mathbf{f}_k & \mathbf{b}_k & \mathbf{f}_a & \mathbf{b}_a \\ \hline \mathbf{f}_f & s & & & & \\ \mathbf{f}_k & & s & & & \\ \mathbf{b}_k & & & s & & \\ \mathbf{f}_a & & & & s+v & \\ \mathbf{b}_a & & & & & s-v \end{array}$$

The impedance tensor is

$$\mathbf{Z}_2 = \begin{array}{c|ccccc} & \mathbf{f}_f & \mathbf{f}_k & \mathbf{b}_k & \mathbf{f}_a & \mathbf{b}_a \\ \hline \mathbf{f}_f & 2(r_f+jsX_f) & jsx_{ad} & jsx_{ad} & jsx_{ad} & jsx_{ad} \\ \mathbf{f}_k & jsx_{ad} & r_{kS}+jsX_{kS} & r_{kD}+jsX_{kD} & jsX_{aS} & jsX_{aD} \\ \mathbf{b}_k & jsx_{ad} & r_{kD}+jsX_{kD} & r_{kS}+jsX_{kS} & jsX_{aD} & jsX_{aS} \\ \mathbf{f}_a & j(s+v)x_{ad} & j(s+v)X_{aS} & j(s+v)X_{aD} & r_a+j(s+v)X_S & j(s+v)X_D \\ \mathbf{b}_a & j(s-v)x_{ad} & j(s-v)X_{aD} & j(s-v)X_{aS} & j(s-v)X_D & r_a+j(s-v)X_S \end{array}$$

$$\mathbf{e}_2 = \begin{array}{ccccc} \mathbf{f}_f & \mathbf{f}_k & \mathbf{b}_k & \mathbf{f}_a & \mathbf{b}_a \\ \hline & & & e_f & \end{array}$$

The tensors $\mathbf{G}_2$ and $\mathbf{i}_2$ are identical with $\mathbf{G}_1$ and $\mathbf{i}_1$. Hence the steady-state voltage equations along the sequence axes are $\mathbf{e}_2 = \mathbf{Z}_2\mathbf{i}_1$, and the torque remains unchanged.

## THE TRANSIENT SEQUENCE EQUATIONS

Since $s$ = slip is the actual frequency $f$ impressed along the axes rotating with the field, in the above matrix (or in the equivalent circuit of Fig. A.1$a$) let:

1. $s$ be replaced by $p/j = -jp$.
2. $v$ be replaced by $p\theta$.
3. $x$ be replaced by $L$.

However, in synchronous-machine theory it is customary to write all $L$ as $x$, even in transient equations. That convention will be retained in this example, although often it leads to confusion.

With these substitutions the transient impedance tensor along the sequence axes becomes

$$\mathbf{Z}_3 = \begin{array}{c|c|c|c|c|c|} & \mathbf{f}_f & \mathbf{f}_k & \mathbf{b}_k & \mathbf{f}_a & \mathbf{b}_a \\ \hline \mathbf{f}_f & 2(r_f+X_fp) & x_{ad}p & x_{ad}p & x_{ad}p & x_{ad}p \\ \hline \mathbf{f}_k & x_{ad}p & r_{kS}+X_{kS}p & r_{kD}+X_{kD}p & X_{aS}p & X_{aD}p \\ \hline \mathbf{b}_k & x_{ad}p & r_{kD}+X_{kD}p & r_{kS}+X_{kS}p & X_{aD}p & X_{aS}p \\ \hline \mathbf{f}_a & (p+jp\theta)x_{ad} & (p+jp\theta)X_{aS} & (p+jp\theta)X_{aD} & r_a+(p+jp\theta)X_S & (p+jp\theta)X_D \\ \hline \mathbf{b}_a & (p-jp\theta)x_{ad} & (p-jp\theta)X_{aD} & (p-jp\theta)X_{aS} & (p-jp\theta)X_D & r_a+(p-jp\theta)X_S \\ \hline \end{array}$$

The transient voltage equations are $\mathbf{e}_1 = \mathbf{Z}_3\mathbf{i}_1$. The torque formula remains unchanged.

The same transient equations are found from Fig. A.1*b*, after multiplication by the transient-frequency tensor $\mathbf{n}$ (containing $-jp$ in place of $s$ and $p\theta$ in place of $v$).

These transient equations contain the operator $j$. In order to get rid of it, it is necessary to introduce a physical reference frame.

## THE TRANSIENT EQUATIONS ALONG THE PHYSICAL d AND q AXES

Let the sequence currents be replaced by the transformation $\mathbf{i}_2 = \mathbf{C}\mathbf{i}_4$, where, by Eq. 3.5,

$$\mathbf{C} = \frac{1}{\sqrt{2}} \begin{array}{c|c|c|c|c|c|} & \mathbf{d}_f & \mathbf{d}_k & \mathbf{q}_k & \mathbf{d}_a & \mathbf{q}_a \\ \hline \mathbf{f}_f & 1 & & & & \\ \hline \mathbf{f}_k & & 1 & j & & \\ \hline \mathbf{b}_k & & 1 & -j & & \\ \hline \mathbf{f}_a & & & & 1 & j \\ \hline \mathbf{b}_a & & & & 1 & -j \\ \end{array}$$

Then the new $\mathbf{Z}_4$ becomes, by $\mathbf{C}^*{}_t\mathbf{Z}_3\mathbf{C}$,

$$
\mathbf{Z}_4 = \begin{array}{c|c|c|c|c|c}
 & \mathbf{d}_f & \mathbf{d}_k & \mathbf{q}_k & \mathbf{d}_a & \mathbf{q}_a \\
\hline
\mathbf{d}_f & r_f + X_f p & x_{ad}p & & x_{ad}p & \\
\mathbf{d}_k & x_{ad}p & r_{kd} + X_{kd}p & & x_{ad}p & \\
\mathbf{q}_k & & & r_{kq} + X_{kq}p & & x_{aq}p \\
\mathbf{d}_a & x_{ad}p & x_{ad}p & -x_{aq}p\theta & r_a + x_d p & -x_q p\theta \\
\mathbf{q}_a & & x_{ad}p\theta & x_{aq}p & x_d p\theta & r_a + x_q p
\end{array}
$$

The new $\mathbf{e}_3$ becomes, by $\mathbf{C}^*{}_t\mathbf{e}_2 = \mathbf{e}_3$,

$$
\mathbf{e}_3 = \begin{array}{ccccc}
\mathbf{d}_f & \mathbf{d}_k & \mathbf{q}_k & \mathbf{d}_a & \mathbf{q}_a \\
\hline
\phantom{x} & \phantom{x} & \phantom{x} & e_f/\sqrt{2} & -je_f/\sqrt{2}
\end{array}
$$

$$
\mathbf{i}_2 = \begin{array}{ccccc}
\mathbf{d}_f & \mathbf{d}_k & \mathbf{q}_k & \mathbf{d}_a & \mathbf{q}_a \\
\hline
i^{fd} & i^{kd} & i^{kq} & i^{ad} & i^{aq}
\end{array}
$$

$$
\mathbf{G}_2 = \begin{array}{c|c|c|c|c|c}
 & \mathbf{d}_f & \mathbf{d}_k & \mathbf{q}_k & \mathbf{d}_a & \mathbf{q}_a \\
\hline
\mathbf{d}_a & & & -x_{aq} & & x_q \\
\mathbf{q}_a & x_{ad} & x_{ad} & & x_d &
\end{array}
$$

The transient voltage equations along the physical axes are $\mathbf{e}_3 = \mathbf{Z}_4\mathbf{i}_2$ and the torque equation is $T = \mathbf{i}_2\mathbf{G}_2\mathbf{i}_2$. These equations do not contain the operator $j$.

During *acceleration*

$$T = \mathbf{i}_2\mathbf{G}_2\mathbf{i}_2 - Ip^2\theta$$

During steady state

$$p = jf = js$$

# APPENDIX 2

# DESIGN CONSTANTS OF THE SHADED-POLE MOTOR

## NEW TYPES OF DESIGN CONSTANTS

In Fig. 5.18*a* the shaded coil was assumed to lie at an angle $\alpha$ from the main axis. The relation between the shaded-pole motor and an equivalent split-phase motor are given in Fig. 5.18*d*. These relations do not contain the angle of shift $\alpha$ and the ratio of turns $a$.

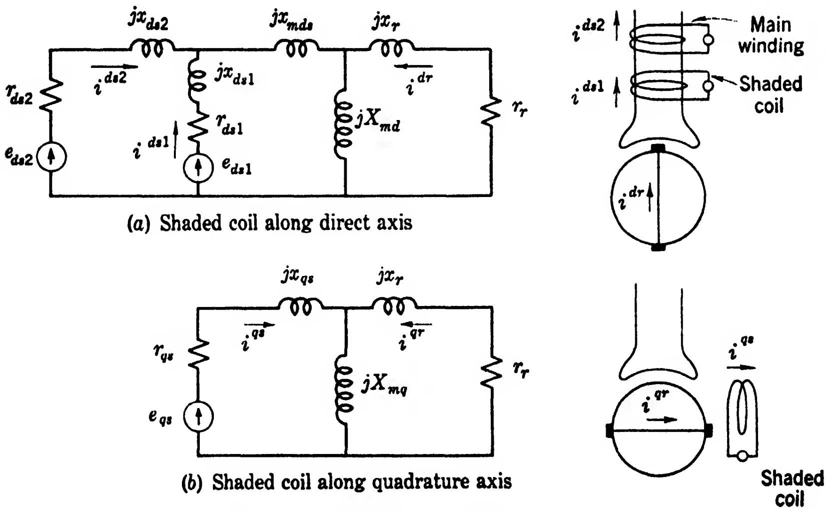



Fig. A.2. Reactances of the shaded-pole motor.

In order to introduce $\alpha$ and $a$, let it be assumed that the shaded coil is placed first along the direct axis ($\mathbf{d}_{s1}$) and then along the quadrature axis ($\mathbf{q}_{s1}$) (Fig. A.2). The impedances of the coil in the two positions

and of the main winding ($\mathbf{d}_{s2}$) are

$$
\begin{array}{cc|c|c|c|}
 & & \mathbf{d}_{s2} & \mathbf{d}_{s1} & \mathbf{q}_{s1} \\
 & \mathbf{d}_{s2} & r_{ds2} + jX_{ds2} & jX_{ms} & \\ \cline{3-5}
\mathbf{Z} = & \mathbf{d}_{s1} & jX_{ms} & r_{s1} + jX_{ds1} & \\ \cline{3-5}
 & \mathbf{q}_{s1} & & & r_{s1} + jX_{qs1}
\end{array}
$$

If the shaded coil is now placed at its true position at an angle $\alpha$, the transformation is $\mathbf{i} = \mathbf{C}_1\mathbf{i}$.

$$
\begin{array}{cc|c|c|}
 & & \mathbf{m} & \mathbf{s} \\ \cline{3-4}
 & \mathbf{d}_{s2} & 1 & \\ \cline{3-4}
\mathbf{C}_1 = & \mathbf{d}_{s1} & & a\cos\alpha \\ \cline{3-4}
 & \mathbf{q}_{s1} & & a\sin\alpha \\ \cline{3-4}
\end{array}
$$

where $i^m$ and $i^s$ represent the actual winding currents.

## REPLACING CURRENTS BY MMF'S

Let next the actual winding currents $i^m$ and $i^s$ be replaced by the resultant mmf's $i^{ds}$ and $i^{qs}$ existing along the direct and quadrature axes by $\mathbf{i}'' = \mathbf{C}_2^{-1}\mathbf{i}'$.

$$
\begin{array}{cc|c|c|}
 & & \mathbf{m} & \mathbf{s} \\ \cline{3-4}
\mathbf{C}_2^{-1} = & \mathbf{d}_s & 1 & a\cos\alpha \\ \cline{3-4}
 & \mathbf{q}_s & & a\sin\alpha \\ \cline{3-4}
\end{array}
\qquad
\begin{array}{cc|c|c|}
 & & \mathbf{d}_s & \mathbf{q}_s \\ \cline{3-4}
\mathbf{C}_2 = & \mathbf{m} & 1 & -\cot\alpha \\ \cline{3-4}
 & \mathbf{s} & & 1/a\sin\alpha \\ \cline{3-4}
\end{array}
$$

The resultant transformation from the original shaded coils to the rectangular mmf variables is

$$
\begin{array}{cc|c|c}
 & & \mathbf{d}_s & \mathbf{q}_s \\
 & \mathbf{d}_{s2} & 1 & -\cot\alpha \\ \cline{3-4}
\mathbf{C} = \mathbf{C}_1\mathbf{C}_2 = & \mathbf{d}_{s1} & & \cot\alpha \\ \cline{3-4}
 & \mathbf{q}_{s1} & & 1
\end{array}
$$

The new impedance tensor is, by $\mathbf{C}_t\mathbf{Z}\mathbf{C}$,

$$\mathbf{Z}' = \begin{array}{c|c|c} & \mathbf{d}_s & \mathbf{q}_s \\ \hline \mathbf{d}_s & r_{ds2} + jX_{ds2} & -(r_{ds2} + jX_{ds2}) \cot \alpha \\ \hline \mathbf{q}_s & -(r_{ds2} + jX_{ds2}) \cot \alpha & r_{qs1} + jX_{qs1} + (r_{ds1} + jX_{ds1} + r_{ds2} + jX_{ds2}) \cot^2 \alpha \end{array}$$

## EQUIVALENT SPLIT-PHASE MOTOR

The impedance term of an equivalent split-phase induction motor (with axes at right angles along $\mathbf{d}_s$ and $\mathbf{q}_s$) and with mutual impedance $Z'_m$ between the $\mathbf{d}_s$ and $\mathbf{q}_s$ axes is

$$Z' = \begin{array}{c|c|c} & \mathbf{d}_s & \mathbf{q}_s \\ \hline \mathbf{d}_s & r'_{ds} + jX'_{ds} + Z'_m & -Z'_m \\ \hline \mathbf{q}_s & -Z'_m & r'_{qs} + jX'_{qs} + Z'_m \end{array}$$

Equating the coefficients of the last two matrices, the following relations exist between the stator constants of the equivalent motor (primed quantities) and the known design constants of the shaded coil, placed first along the $\mathbf{d}$, then along the $\mathbf{q}$ axis.

$$\begin{aligned} r'_{ds} &= r_{ds2}(1 - \cot \alpha) \\ X'_{ds} &= X_{ds2}(1 - \cot \alpha) + X_{dsm} \\ r'_{qs} &= r_{qs1} + r_{ds1} \cot^2 \alpha + r_{ds2} \cot \alpha(\cot \alpha - 1) \\ X'_{qs} &= X_{qs1} + X_{ds1} \cot^2 \alpha + X_{ds2} \cot \alpha(\cot \alpha - 1) \\ Z'_m &= (r_{ds2} + jX_{ds2}) \cot \alpha \end{aligned}$$

The relations among the currents and voltages are

$$i^m = i^{ds} - i^{qs} \cot \alpha \qquad e_{ds} = e_m$$

$$i^s = i^{qs}/a \sin \alpha \qquad e_{qs} = e_s/(a \sin \alpha) - e_m \cot \alpha$$

*These new relations containing the angle of shift $\alpha$ and the ratio of turns $a$ take the place of the relations in Fig. 5.18d.* The equivalent circuits of Figs. 5.19, 5.20, 10.13, and 10.14 are still valid.

# APPENDIX 3

# VISUALIZABLE AND NON-VISUALIZABLE PHYSICAL VECTORS

## UNDERLYING SPACES AND TANGENT SPACES

The description of a physical phenomenon in terms of partly visualizable and partly non-visualizable vectors (Chapter 1) seems to be a general trend in modern scientific thought. To emphasize the significance of this trend, a more general *geometrical* model will now be established, in which even the hitherto non-visualizable physical vectors appear as geometrical vectors that can be visualized.

Let a curved space be given, say a spherical surface, and let exist at each point on the surface a physical vector, such as a flux-density vector. In order to represent the magnitude and direction of each physical vector by some geometrical vector, it is assumed that *at each point of the sphere a tangent plane exists* and the vectors lie at each point in the corresponding tangent plane (Fig. A.3).

In general, it is assumed that at each point of an $n$-dimensional curved "underlying" space, an $n$-dimensional flat "tangent" space is superimposed, so that a vector at any point of the underlying space also lies in the tangent or "local" space at that point. In the tangent space every vector starts at the point of contact with the underlying space; hence a tangent space is also called a "centered" space. When the point of contact moves, its instantaneous velocity and acceleration vectors, together with their tangent or "local" spaces, also move.

Now *in rotating machinery the instantaneous charges and displacements represent the coordinates of a point in the underlying space.* Since the former have disappeared from view, the underlying space is also hidden. But the moving tangent "local" spaces, containing the radial vectors

**I**, **B**, **E**, etc., can be visualized at each instant and, from the peculiar behavior of these vectors in the local space, the existence of an underlying space may be inferred.

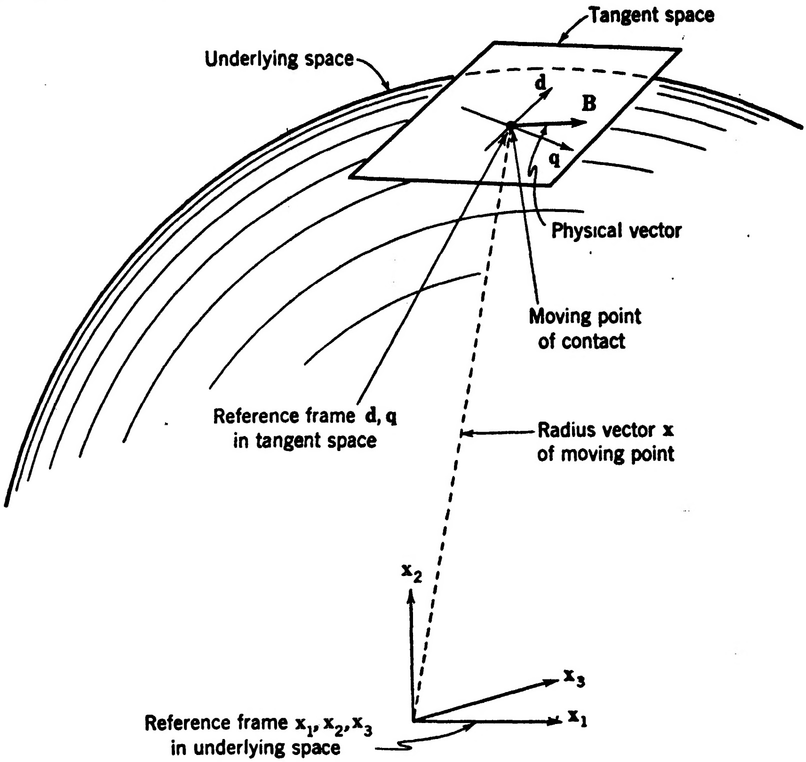



Fig. A.3. Underlying space and moving tangent space.

## FORMAL ANALOGIES WITH BASIC PHYSICS

*In our actual physical universe similar phenomena exist*, as represented by the equations of the various "unified field theories." In these theories also, only the "local" three-dimensional Euclidean space (and time) can be visualized, whereas the existence of an "underlying" *curved* space-time can be inferred only from the peculiar behavior of the physical vectors visualized in the local spaces.

As was mentioned before, the equations of the various "unified field theories" bear striking *formal* analogies to the equations of rotating electric machinery, when the latter are formulated in tensorial (or "invariant") form. The analogy not only extends to the coexistence of electromagnetic and mechanical (gravitational) energies mentioned in the introduction and to the appearance of visualizable local and non-

visualizable underlying spaces but also includes many other concepts, whose treatment is out of place here. One other analogy (the "cylindrical" properties of the variables) has been mentioned in the Epilogue.

Attention is called to these formal analogies here merely to emphasize the need for only a strikingly few basic *physical concepts* in order to study the behavior of the large variety of electrical engineering structures. Moreover, these few basic laws are not peculiar to engineering. It is possible to formulate these foundations of engineering science in such a systematic manner that the engineer can pass easily from the analysis of man-made engineering systems to the study of edifices constructed by nature. The laws of expanding galaxies and whirling electrons and rotating electric machines differ only in their minute details, not in their broad outlines.

Every attempt has been made in this book (as in every other book of the author) to introduce only the physical concepts that form a part of modern physics. At the same time no effort was spared to introduce only as few of these basic concepts as is necessary for the problem at hand. With fewer and simpler concepts, progress is much faster and more penetrating.